National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

NL-339 (r. 88/04) c

Canada

THE UNIVERSITY OF ALBERTA

THERMO FIELD DYNAMICS AND RADIATIVE CORRECTIONS IN QUANTUM ELECTRODYNAMICS

BY

SANJAY CHUGH

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

IN

THEORETICAL PHYSICS

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

SPRING 1989

National Library of Canada
Bibliothèque nationale du Canada

Canadian Theses Service
Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-52971-7

Canada

THE UNIVERSITY OF ALBERTA
RELEASE FORM

NAME OF AUTHOR:	Sanjay Chugh
TITLE OF THESIS:	Thermo Field Dynamics and Radiative Corrections in Quantum Electrodynamics
DEGREE:	Master of Science

YEAR THIS DEGREE GRANTED: 1989

Permission is herby granted to THE UNIVERSITY OF ALBERTA LIBRARY to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

Sanjay Chugh
(Student's signature)

240 TUNDRA DR.
(student's permanent address)
Fort McMurray, AB T9H 4S8

THE UNIVERSITY OF ALBERTA
FACULTY OF GRADUTE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled THERMO FIELD DYNAMICS AND RADIATIVE CORRECTIONS IN QUANTUM ELECTRODYNAMICS submitted by SANJAY CHUGH in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE in THEORETICAL PHYSICS

Supervisor

Date: 21 April, 1989

To My Parents

Abstract

Thermo field dynamics, which is a quantum field theory for finite temperature is very useful for practical calculation because the Feynman diagram method is retained. We discuss in detail the various aspects of equilibrium thermo field dynamics, showing how the results of the usual quantum field theory can be rewritten in terms of the thermal doublet notation. We then use the thermo field dynamics to recast the results of quantum electrodynamics, discussing in particular the electron and photon self-energy and the vertex diagram. At the end we discuss how these changes effect the radiative corrections to Coulomb scattering, and apply the results to the famous Lamb shift. This is a very interesting calculation in light of the very accurate experimental measurements of the Lamb shift.

ACKNOWLEDGEMENTS

I thank Dr. H. Umezawa for introducing me to Thermo Field Dynamics, and presenting me with a very interesting problem. I also thank Dr. Y. Yamanka for helping me learn Quantum Electrodynamics, and for helping, and guiding me all through my work.

Table of Contents

Chapter Page

Introduction 1
Unitary Equivalent and Inequivalent Representations 2
The Dual Structure of QFT 3
Thermal Degrees of Freedom 4
Matsubara Formalism 5
Introduction to the Lamb Shift 7
Vacuum Polarization 9
Electron-Self Energy 10
Plan of the Thesis 13
Thermo Field Dynamics 15
The Thermal Average and the Tilde Field 15
The Density Matrix Method 16
Fundamentals of TFD 22
The Hamiltonian and Momentum 22
The Bogoliubov Transformation 22
Determination of θk 24
The Tilde Operation 25
The Heisenberg Equation 26
The Thermal State Condition 28
The Heisenberg Equation, the Dynamical Map and the Kubo-Martin-Schwinger Equation 31
The Heisenberg Equation 31
The Dynamical Map 32
Two-Point Functions in TFD 35
Examples of Two Point Functions in TFD 35
Fermion System 35
Boson System 37
The Klein-Gordon Field 39
Spectral Representation of Two-Point Functions 40
Single Particle States 45
Analytic Properties of the Causal Green's Function 47
Bethe-Salpeter Equations at Finite Temperature 51

The Feynman Propagators 53
The Photon Propagator 54
The Electron Propagator 55
The Electron Self-Energy, Vacuum Polarization and the Vertex 61
Feynman Rules 61
The Electron Self-Energy 64
The Vacuum Polarization 70
The Vertex 73
Electron Propagator in External Field 84
Radiative Corrections to Coulomb Scattering and the Lamb Shift 92
Radiative Corrections to Coulomb scattering 92
The Lamb Shift 95
Summary 108
Bibliography 110
References 112
Appendix A
Free Fields for Physical Particles 115
Appendix B
Product Rules of Two Point Functions 121
Appendix C
Thermal Separation of Propagators 124
Appendix D
Notation 126

List of Tables

Table	Description	Page
1-1	Low-lying energy levels of atomic hydrogen	7
11-1	Feynman rules for spinor electrodynamics in TFD	63

List of Figures

Figure	Description	Page
1.1	Low lying energy levels of atomic hydrogen	9
1.2	Effect of vacuum polarization on the charge of the electron	10
2.1	Absorption of energy by a system in a reservoir	20
11.1	Lowest order electron self-energy	64
11.2	Electron self-energy parts inserted into a Compton Diagram	64
11.3	Electron mass counterterm	65
11.4	Lowest order photon self energy	70
11.5	Photon self-energy inserted into a Compton diagram	70
11.6	Second order vertex diagram	73
11.7	Vertex parts inserted into a Compton diagram	73
12.1	Modification of Feynman diagram in presence of external field	90
12.2	Indicating action of external field by a Feynman diagram	91
13.1	Coulomb scattering	92
13.2	Radiative corrections to Coulomb scattering	93
13.3	Fluctuation and Polarization diagrams for the Lamb shift	96
13.4	Expansion of the fluctuation diagram in powers of the external field	98
13.5	Extracting the free electron self-energy from the fluctuation diagram	99

Introduction

The early part of this century saw a revolution in physics, the likes of which had never occurred before. Today this is known as the quantum revolution. It ended with the formulation of quantum mechanics (QM) by 1925 [1]. This was followed by many successful applications of the theory to physical phenomena. The final form of the theory, as known today, was the work of Dirac.

Although the theory did have many successes, it also had weaknesses; it was not Lorentz invariant. However, there were many new developments by 1935, the year of the famous paper by Yukawa. These new developments in relativistic quantum field theory(QFT) were initiated by the Yukawa meson theory which opened the new high energy physics with many particles.

The relationship between QFT and classical field theory is analogous to that between QM and classical mechanics. The classical field quantities are replaced by noncommutative objects referred to as the field operators* . During the reconciliation of QM and special relativity, two major differences arose between QM and QFT [2] :

1) in QM, the position c-number **x** is replaced by the operator **q**, whereas QFT preserves the position.

2) quantum mechanics treats systems with a finite number of degrees of freedom and QFT treats systems with an infinte number of degrees of freedom.

This second difference between the two theories is a very significant one. In QM, one always gets the same answer to a physical problem, but this need not be the case in QFT.

---

* Particles that obey Bose-Einstein statistics satisfy commutation relations, whereas particles that obey Fermi-Dirac statistics satisfy anti-commutation relations.

Unitary Equivalent and Inequivalent Representations

When working on a problem in QM, one has several choices of representations in which to solve the problem. For example, a problem might be solved with less effort in the coordinate representation then in the harmonic oscillator representation. However, once the problem has been solved, it has in essence been solved in any representation. This is because in QM it is possible to change from one representation to another through a unitary transformation. Thus the representations in QM are referred to as unitarily equivalent representations. Hence the answer to a physical problem is always the same in QM , independent of the representation.

The situation is quite different in QFT. The representations in QFT are frequently unitarily inequivalent* [3,4] in the sense that a vector in one representation cannot be a superposition of basic vectors in another representation. Hence, it is not possible to change from one representation to another through a unitary transformation. It is possible to get different answers to the same physical problem just by solving it in different representations. This apparent malady is a direct consequence of the fact that QFT treats systems with infinitely many degrees of freedom. This infinity is not just 'very large' as one might think, but is in fact genuine infinity.

To understand this, consider a system of fermions. The states of the system are labelled by i = 0,1,2,... . We consider fermions because the particle number in the i-th state can only be 0 or 1. If the particle number is specified by n_i then the state of a many body system can be specified by a sequence of all n_i. Thus for example, 10011... may represent such a state. This is equivalent to the binary real number 0.10011... . Because the set of such numbers is not countable, neither is the set of states being considered.

---

* The word unitary equivalence in quantum mechanics is used in two different ways : one is the existence of unitary mapping (or correspondence) between two representations and another is the equivalence through the superposition of vectors. Here we are referring to the latter definition.

Using superposition of these state vectors, other sets of orthogonal state vectors may be constructed. However, the normalizability of each state vector is something that is essential for the theory to be a physical theory. This requires the Hilbert space to be separable* and the set of the basic orthogonal vectors to be countable. This is the same as saying, that the probability concept must be preserved. Thus, it is that a countable set is needed. This can be chosen from the uncountable sets. However, the question then asked is how to choose this set since there are many possibilities. These different sets correspond to the different representations and so we arrive at unitary inequivalence.

Although, this may seem to be a problem at first, it is in fact quite advantageous especially in condensed matter physics. Thus the different countable subsets correspond to the many phases exhibited by many body systems; for example, normal conductivity and superconductivity are described by different subsets.

The Dual Structure of QFT

During the development of QFT, it was essentially an analytical method for the physics of elementary particles. However, it soon became clear that QFT also supplies a language for the description of quantum many-body systems. This was chiefly due to the development of the Fock representation. This representation classified the states of a quantum system by a number series ; for example as explained earlier for a system of fermions. In light of this, it is clear that just such a representation is required for the description of a many-body system.

Quantum field theory was initially formulated in terms of the interacting Heisenberg fields and observable free fields. The basic relations were expressed in terms of the

---

* A space $\mathfrak{h}$ is said to be separable, if it contains a countable basis $\{\xi_n\}$ such that any vector ξ in $\mathfrak{h}$ can be approximated by a linear combination of ξ_n (i.e. $\Sigma c_n\xi_n$) to any accuracy. In other words, for every ξ in $\mathfrak{h}$ and any $\varepsilon > 0$ there exists a sequence $\{c_n\}$ such that $|\xi - \Sigma_n c_n\xi_n| < \varepsilon$ for arbitrary ε.

Heisenberg fields, but the theoretical results were expressed in terms of the free fields. In condensed matter physics for example, some of these free fields are the phonons, magnons, plasmons etc. Hence it is realized that QFT has a dual structure and that to solve a problem, one need only find a mapping between the Heisenberg fields and the incoming fields. This mapping is referred to as the dynamical map. The dynamical map is a generalization of what is called Haag expansion or the LSZ expansion [5] in zero temperature QFT. In this case the free fields are nothing else then the asymptotic fields; still the word, dynamical map, is used even with non-zero temperature field theory. The dynamical map often becomes very complicated. For example, even if the Lagrangian has a symmetry, its phenomonological appearance may lose this symmetry because the dynamical map leads to the spontaneous breakdown of symmetry.

Thermal Degrees of Freedom

It might be obvious to think that since every object has a finite size, then it should have a finite number of degrees of freedom. However this is usually not true. For example, it is well known that the hydrogen atom has infinitely many degrees of freedom because the electron and proton interact with the electromagnetic field (i.e through the Lamb shift). Furthermore, a finite sized system has a free boundary surface which contains surface waves such as the surface phonon. Thus it quite possible for a finite system to have infinitely many degrees of freedom. As already discussed, such a system has many choices for the state vector space. These choices include solutions which describe the coexistence of classical and quantum objects by creating classical objects through boson condensation. This, although being advantageous, is also a weak point in QFT. This weakness results in the lack of uniqueness in the solution for a given problem. So it becomes obvious that additional degrees of freedom are required. These degrees of freedom are the thermal degrees of freedom.

On the other hand a quantum field theory with thermal degrees of freedom is expected to be self contained, because it is capable of describing many phases without sacrificing uniqueness. Such a theory can derive classical phenomena from quantum origin as well as describing transitions among different phases. Hence QFT must be revised to include the effects of temperature. The first attempt at the formulation of a field theory at finite temperature was by Matsubara.

Matsubara Formalism

In statistical mechanics, the important quantity is the average of a quantity, say A for example. This is given by

$$\langle A \rangle = \frac{1}{Z(\beta)} \mathrm{Tr}[A e^{-\beta h}] \tag{1.5.1}$$

where

$$h = H - \mu N \tag{1.5.2a}$$

$$Z(\beta) = \mathrm{Tr}[\, e^{-\beta h}\,] \tag{1.5.2b}$$

$$\beta = \frac{1}{k_B T} \tag{1.5.2c}$$

and μ is the chemical potential. The quantity $Z(\beta)$ is the grand canonical partition function. In order to apply QFT to many body problems, an analogy must be developed between QFT and statistical mechanics. This analogy was discovered by Matsubara in 1955 [6] . He was able to formulate the calculation of the vacuum expectation value of the operator A in terms of the theory of the interaction representation which was developed originally in QFT. He then developed a remarkable method for computing the partition function in which the Feynman diagram method could be applied.

Later it was found by Abrikosov et al. [7] and Umezawa et al. [8] that the Matsubara theory can be formulated in terms of the Feynman diagram method with discrete complex energies, the so called Matsubara frequencies. These appear both on internal and

external lines of Feynman diagrams. The internal energies must be summed. This summation has several available prescriptions. The external energies define the Green's function at a discrete set of points in the complex energy plane. This leads to a difficulty because for answering dynamical questions, a knowledge of Green's functions with real continuous energy (i.e. real time Green's functions) is required. Hence the Green's functions in the Matsubara formalism must be analytically continued from the complex plane onto the real axis. Although this is feasible, it is quite difficult in the case of several external energies. Furthermore, the analytic continuation is not unique even in the case of a single external energy. In addition, many properties of operator formalism such as the dynamical map and Ward-Takahashi identities are difficult. Thus a finite temperature theory with full use of quantum field theory with real time is desired.

In 1971, Umezawa, Mancini and Leplae [9] presented a method which also allowed the use of the Feynman diagram technique. They used this method in the construction of quantum field theoretical formulation for superconductivity with an effective use of an operator canonical transformation which easily treats the Coulomb potential effect. This method was put in a systematic form by Takahashi and Umezawa [10] in 1974. Unlike the Matsubara formalism however, their method was the real time finite temperature field theory. Takahashi and Umezawa named it thermo field dynamics (TFD). Even today, with the existence of several formulations for field theories at finite temperature, TFD remains one of the most useful for practical calculations.

With the increasing popularity of TFD, and the very accurate measurements of the Lamb shift, we are motivated to calculate the temperature corrections for the Lamb shift. Below we give an introductory account of the Lamb shift.

Introduction to the Lamb Shift

In the Dirac theory, the energy levels of a hydrogen like atom (i.e. for an electron in a pure Coulomb field) with nuclear charge $-Ze$ depend on the principle quantum number n and the total angular momentum j. The first few low-lying energy levels are listed in table 1-1.

Table 1-1 : Low-lying energy levels of atomic hydrogen

	n	l	j	E_{nj}
$1S_{1/2}$	1	0	1/2	$m\sqrt{1-Z^2\alpha^2}$
$2S_{1/2}$	2	0	1/2	$m\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{1/2}$	2	1	1/2	$m\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{3/2}$	2	1	3/2	$\frac{m}{2}\sqrt{4-Z^2\alpha^2}$

From this table it is seen that the $2S_{1/2}$ and $2P_{1/2}$ states are degenerate, being the two eigenstates of opposite parity corresponding to the same n and j. Also the $2P_{3/2}$ state is higher in energy then the $2P_{1/2}$ state; the difference being the fine-structure splitting due to the spin orbit interaction.

When the above predictions were modified to take into account the hyperfine splitting of each level due to coupling between the electron and proton spins, they were in complete satisfactory agreement with experimental data prior to 1947. In 1947, Lamb and Retherford reported a shift of the $2S_{1/2}$ levels upward relative to the $2P_{1/2}$ levels.* [11] This shift subsequently referred to as the "Lamb shift" breaks the degeneracy of levels with the same n and j but different l and arises from the interaction of the electrons with the

* Lamb and Retherford measured the $2P_{3/2}$ - $2S_{1/2}$ transitions as a function of an applied magnetic field.

fluctuations of the quantized radiation field (radiative correction). This is illustrated in fig. 1.1.

The experiment of Lamb and Retherford was the first decisive one to measure the Lamb shift in the hydrogen spectrum. However prior to this experiment, spectroscopic methods had indicated an anamoly in the hydrogen fine structure. This had been interpreted by Pasternack [12] in terms of an upward displacement of the $2S_{1/2}$ level by 0.03 cm^{-1} (approximately 1000 MHz), due to a repulsive deviation from the Coulomb potential. Also Uehling [13] had shown earlier that vacuum polarization had the effect of displacing the nS levels for an electron moving in a hydrogen-like atom. The displacement is

$$\Delta E(nS) = \frac{-8Z^2\alpha^3}{15\pi n^3} R \qquad (1.6.1)$$

where R is the Rydberg constant. For hydrogen and n=2, the frequency displacement is

$$\Delta\nu = -27 \text{ MHz}$$

This is of opposite sign and about 40 times smaller then the Lamb shift, so it is not the major contributor to the Lamb shift. It is an important effect however and is discussed below.

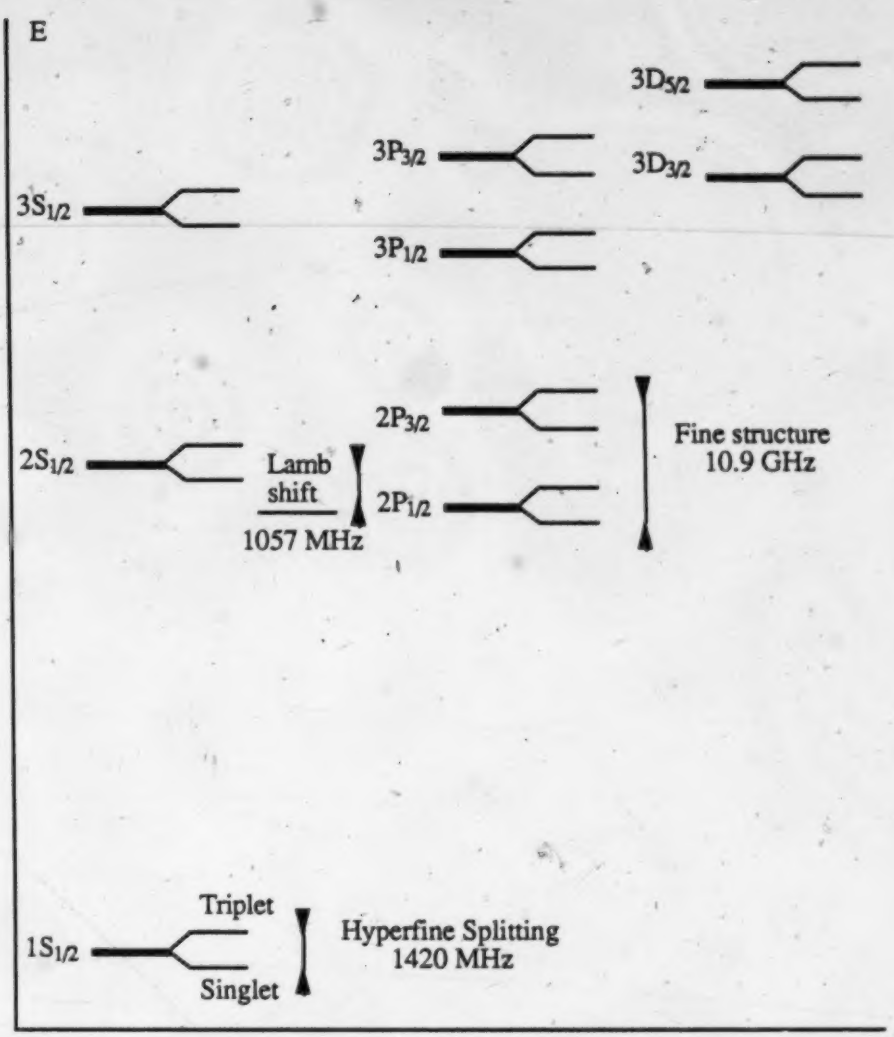


Fig. 1.1 Low-lying energy levels of atomic hydrogen (not to scale)

Vacuum Polarization

We consider the influence of the vacuum on the definition of the charge and upon the interaction between two charges, in terms of the Dirac hole theory. Thus a positive-energy electron (positron) electrostatically repels the electrons in the negative energy sea, thereby polarizing the vacuum in its vicinity. The charge density of the electron $\rho_0(\mathbf{r})$ plus the polarized vacuum $\rho_P(\mathbf{r})$ measured relative to the vacuum is illustrated in fig 1.2. The

test charge at a large distance is $\int d^3r\,[\rho_0(\mathbf{r}) + \rho_p(\mathbf{r})] = e$ which is the physical charge. If a test charge is probing at distances $r_0 < R$ the apparent charge becomes more negative until as $r_0 \to 0$ the charge becomes $\int d^3r\,\rho_0(\mathbf{r}) = e_0$ which is referred to as the "bare" charge and $|e_0| > |e|$. This is why we get a shift in the hydrogen atom spectrum from vacuum polarization. In other words the s levels are lowered relative to those with angular momentum $l \neq 0$, since the $l = 0$ wave functions bring electrons close to the protons.

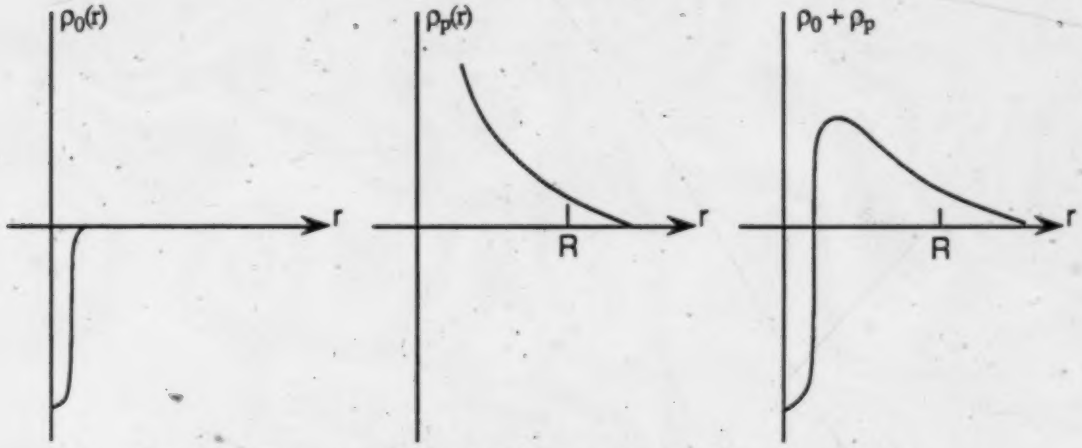


Fig. 1.2 Effect of vacuum polarization on the electrons charge density. ρ_0 is the charge density of the "bare" electron and ρ_p that of the induced polarization "cloud" of virtual electron-positron pairs.

In addition to the vacuum polarization (self-energy of the photon), the radiative corrections are also due to the self-energy of the electron, and this is discussed below.

Electron-Self Energy

Before the advent of quantum mechanics, the electron when considered classically to be a particle of charge e mass m then its radius a was given by

$$a = \frac{e^2}{mc^2} \qquad (1.6.2)$$

This expression is obtained from the assumption that the electrostatic energy of an electron at rest (i.e. the electrostatic self-energy) accounts fully for its mass : $e^2/a = mc^2$. The difficulty with this result is that if the radius is not zero then the stress of electromagnetic field will tend to explode this charge distribution. On the other hand, if $a = 0$ then the self-energy is infinite. This problem persisted in quantum mechanics where it was noted that the self energy behaves as the limit e^2/a for $a \to 0$, just as in the classical case.

The situation became worse in quantum field theory where there is in addition to the electrostatic self-energy, the electromagnetic self-energy due to the interaction of the electron with the radiation field. The electromagnetic self-energy can only be calculated by use of pertubration theory. For example, for a free electron with momentum **p** and energy $E(p) = (p^2 + m^2c^2)^{1/2}$ the self energy $W(\mathbf{p})$ is found to be [14]

$$W(\mathbf{p}) = \frac{e^2h^2c^2}{hcE(p)} \int k\, dk \qquad (1.6.3)$$

This expression is quadratically divergent whereas the classical result is linearly divergent. Oppenheimer [15] had done a similar calculation for an electron in a bound atomic state and was led to conclude that self-energy effects cause infinite displacement of spectral lines.

The situation was very critical at this time. So it was that Bethe presented a simple calculation for the Lamb shift that was in remarkable agreement with observation [16]. It was here that the idea of renormalization first became concrete, although the term and the formal procedures did not originate till much later. Bethe's calculation for the nS level shift treated the electron non-relativistically and his argument went like this : the leading term in the electron's self energy diverges linearly as e^2/a. This is the same for a free electron as for a bound electron with the same average kinetic energy. Subtracting this term (mass renormalization) and identifying the remainder as the level displacement gives

$$\Delta E(nS) = \frac{4Z^4\alpha^5}{3\pi n^3} mc^2 \ln\frac{K}{\bar{E}} \qquad (1.6.4)$$

where $\bar{E}$ is the average excitation energy for the nS state. In the limit $K\to\infty$, this result is still infinte. This is where Bethe showed great insight by saying that the subtraction mentioned above "...would set an effective upper limit of mc^2 to the frequencies of light which effectively contribute to the shift of the level of a bound electron." Thus setting $K = mc^2$ gave 1040 MHz for the level shift "in excellent agreement with the observed value".

In addition to Bethe's result, Welton [17] had given a very interesting qualitative description of the Lamb shift. Welton treats the electron non-relativistically in its interaction with the vacuum fluctuations of the electromagnetic field. The calculation is based on the following reasoning : Since the dynamics of a normal mode of the electromagnetic field is equivalent to that of a harmonic oscillator, each mode upon quantization acquires a zero point energy of $\omega/2$. This gives the result that there are now fluctuating electromagnetic fields even when no external fields are applied. Now the average field strengths are zero but the mean square values of the field strengths are not. These mean square values give a mean square value for the electron's position coordinate due to its coupling with the electromagnetic field. The amplitude of this "jiggling" is what Welton estimated. It implies an additional interaction energy $\frac{1}{6}\langle(\delta r)^2\rangle|\psi_n(0)|^2$ from the smearing out of the Coulomb potential V(r) seen by the electron. This gives a shift, to lowest order of

$$\Delta E_n(\text{Lamb}) = \frac{2\pi}{3} Z\alpha\langle(\delta r)^2\rangle|\psi_n(0)|^2 \qquad (1.6.5)$$

The expectation value $\langle(\delta r)^2\rangle$ is approximated to be

$$\langle(\delta r)^2\rangle = \left(\frac{2\alpha}{\pi}\ln\frac{1}{Z\alpha}\right)\left(\frac{1}{m}\right)^2 \qquad (1.6.6)$$

which then gives

$$\Delta E_n = \left[\frac{8}{3\pi}\frac{Z^4\alpha^3}{n^3}\left(\ln\frac{1}{Z\alpha}\right)\right](1/2\ \alpha^2 m)\delta_{l0} \qquad (1.6.7)$$

For a hydrogen atom with $n = 2$, $Z = 1$, $l = 0$, (1.6.7) gives a level shift of 1000 MHz accounting for most of the measured shift of the $2S_{1/2}$ level in the hydrogen atom.

Plan of the Thesis

In the next few chapters, we first give a comprehensive review of equilibrium thermo field dynamics. In chapter 2 we introduce the tilde-field, and discuss its physical interpretation. Also we show how the density matrix formalism may be incorporated into TFD. Chapter 3 discusses the fundamentals of TFD where we list the thermal conjugation rules that relate the tilde-operators to the non tilde-operators. Also we introduce the Bogoliubov transformation for the annihilation and creation operators of the thermal vacuum. In chapter 4 we discuss the thermal state condition, which is fundamental to TFD allowing for the construction of generalized annihilation operators. Chapter 5 shows how the Heisenberg equation can be recast in the thermal doublet notation. Also the dynamical map and the Kubo-Martin-Schwinger equation are discussed. In the next chapter, chapter 6, two-point Green's functions are discussed. A few common examples are given. The following chapter discusses the spectral representation of the two-point function, and the chapter after presents their analytic properties. The final chapter on the review of equilibrium TFD discusses the Bethe-Salpeter equations.

The remaining part of the thesis presents quantum electrodynamics in the context of TFD. First we derive the electron and photon propagator, and separate them into a part which is independent of temperature and a part which is dependent on temperature. In this way the results of the usual QFT can be put into the form of the zero temperature part plus the finite temperature corrections. Hence, we then present the results of the self-energy of the electron and the photon, and also the vertex function. The next chapter discusses the

electron propagator in the presence of the external field, especially its non-relativistic form which is useful for the evaluation of the Lamb shift. The following chapter discusses the radiative corrections to Coulomb scattering and a calculation of the finite temperature correction to the Lamb shift.

Thermo Field Dynamics

The Thermal Average and the Tilde Field

It has been known among axiomatic field theorists that a self-consistent field theory at finite temperature requires the doubling of the degrees of freedom. This was true also in thermo field dynamics, which introduced a fictitious field denoted by a tilde, i.e. $\tilde{A}$ for example. In this way, every operator A has its associated tilde operator $\tilde{A}$ and so the number of degrees of freedom is doubled. In so doing, the statistical average of a quantity A, was expressed by the vacuum expectation value of the operator A.

The starting point for thermo field dynamics was to define a temperature dependent vacuum state $|0(\beta)\rangle\rangle$. This state relates the expectation value of a quantity F to the average in the grand canonical ensemble.

$$\langle\langle 0(\beta)|F|0(\beta)\rangle\rangle = \frac{1}{Z(\beta)} \sum_n \langle n|F|n\rangle e^{-\beta E_n} \tag{2.1.1}$$

In addition, we have the relations $H|n\rangle = E_n|n\rangle$, $\langle n|m\rangle = \delta_{nm}$. This state which is called the thermal vacuum cannot be constructed if the original Hilbert state is retained. This can be seen by expanding the vacuum in terms of the state $|n\rangle$ as

$$|0(\beta)\rangle\rangle = \sum_n |n\rangle f_n(\beta) \tag{2.1.2}$$

From this we immediately get

$$f_m^*(\beta) f_n(\beta) = \frac{1}{Z(\beta)} e^{-\beta E_n} \delta_{nm} \tag{2.1.3}$$

This is not possible if the $f_n(\beta)$ are simply c-numbers. However, the expression is reminiscent of an orthogonality condition in a Hilbert space in which the $f_n(\beta)$ are vectors. Such a representation is realized when an additional fictitious dynamical system is

introduced. All quantities associated with this system are denoted by a tilde. The new fictitious system is characterized by the Hamiltonian $\tilde{H}$ and the vectors $|\tilde{n}\rangle$ so that

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle \tag{2.1.4}$$

$$\langle n|m\rangle = \delta_{nm} \tag{2.1.5}$$

In addition, the following definition is made

$$\langle m|F|n\rangle = \langle \tilde{n}|\tilde{F}^{\dagger}|\tilde{m}\rangle \tag{2.1.6}$$

This definition proves to be very convenient in the development of TFD. Thus, we can write

$$f_n(\beta) = |\tilde{n}\rangle\, e^{-\beta E_n/2}\, Z^{-1/2}(\beta) \tag{2.1.7}$$

so that the thermal vacuum is given by

$$|0(\beta)\rangle = Z^{-1/2}(\beta)\sum_n e^{-\beta E_n/2}|n,\tilde{n}\rangle \tag{2.1.8}$$

Thus the relation (2.1.1) is satisfied. Recently, it was found that the thermo field dynamics can overcome some of the difficulties which arise in the density matrix formalism when a thermal situation is changing in time. Here too, the tilde field plays a decisive role. This is explained below.

The Density Matrix Method

In the language of the density matrix ρ, the basic relations of statistical mechanics are given by

$$i\frac{\partial}{\partial t}\rho = [H, \rho] \tag{2.2.1}$$

$$\langle A\rangle = \frac{Tr[\rho A]}{Tr[\rho]} \tag{2.2.2}$$

The first equation is just the familiar Liouville equation and the second equation is the thermal average. The trace is expressed by

$$Tr[A] = \sum_m \langle m|A|m\rangle \qquad (2.2.3)$$

Now, it will be remembered from the earlier discussion that for systems with infinitely many degrees of freedom, there are infinitely many possible choices for the basic set <ml. This choice gave the freedom to express the different phases of a system by different choices of the basic set. For example, $\{|m_1\rangle\}$, $\{|m_2\rangle\}$ could represent two phases of a system. When this system undergoes a transition from the first phase to the other, then we should be starting with $|m\rangle_1$ and ending up with $|m\rangle_2$. This would imply that the basic set should be dependent on time as denoted by $|m\rangle_t$.

In this way, the trace (2.1.3) becomes time dependent as well and is written as

$$Tr_t[A] = \sum_m {}_t\langle m|A|m\rangle_t \qquad (2.2.4)$$

This time-dependence introduces a complication. However, this can be overcome if statistical mechanics is translated into QFT. We now show that this translation applied to the Liouville equation again requires the doubling of the degrees of freedom. In making this translation, two operations are distinguished. The first is operating on the left of the density matrix ρ as $A\rho$ and the second is operating on the right side of ρ as ρA. The latter operation is considered the same as the left operation on ρ by the tilde conjugate of the operator A as denoted by $\tilde{A}\rho$ (i.e. $\rho A = \tilde{A}\rho$. In the density matrix formalism these doubled operators were called the superoperators [18]. Thermo field dynamics treats superoperators as ordinary operators. Hence the doubling of the degrees of freedom emerges. With this distinction, H in the Liouville equation is replaced by $\hat{H} = H - \tilde{H}$* [19].

---

* In a standard formulation of non-equilibrium statistical mechanics, $\hat{H}$ is denoted by L, the Liouvilian.

Thus if one considers the harmonic oscillator with annihilation and creation operators a and $a^\dagger$ satisfying

$$[a\,,\,a^\dagger]_\sigma = 1 \tag{2.2.5}$$

where $\sigma = 1$ for bosonic and -1 for fermionic oscillators respectively, then when the degrees of freedom are doubled, new hidden oscillator variables appear, $\tilde{a}$ and $\tilde{a}^\dagger$, which satisfy

$$[\tilde{a}\,,\,\tilde{a}^\dagger]_\sigma = 1 \tag{2.2.6}$$

The tilde operators commute with the non-tilde operators. The number operators are thus given by

$$n = a^\dagger a \tag{2.2.7a}$$

$$\tilde{n} = \tilde{a}^\dagger \tilde{a} \tag{2.27b}$$

Now, to express the average of A by the vacuum expectation value, let M stand for a complete set of hermitian mutually commuting observables and let us denote its tilde conjugate by $\tilde{M}$. Their corresponding eigenvalues are denoted by m and $\tilde{m}$, respectively. It should be noted that the sets {m} and {$\tilde{m}$} consist of common elements. So now, if A has only non-tilde operators then

$$\langle m_1, \tilde{m}|A|m_2, \tilde{m}'\rangle = \langle m_1|A|m_2\rangle\delta_{mm'} \tag{2.2.8}$$

From these expressions, we define

$$|0\rangle = \rho^\alpha \sum_m |m, \tilde{m}\rangle \tag{2.2.9a}$$

$$\langle 0| = \sum_m \langle m, \tilde{m}|\rho^{1-\alpha} \tag{2.2.9b}$$

Thus the average of A is given by

$$\langle 0|A|0\rangle = \mathrm{Tr}\,\rho^{1-\alpha}A\rho^\alpha = \mathrm{Tr}\,\rho A \tag{2.2.10}$$

for any value of α. Now, if we call (2.2.9a,b) the vacuum states then the thermal average has been expressed by the vacuum expectation value. So with this doubling of the degrees

of freedom, both the thermal average and the Liouville equation in statistical mechanics are formulated in terms of QFT.

The question to be asked at this point is : what is the physical significance of the tilde field? As we will soon see, it can be shown that the creation of a physical particle is equivalent to the annihilation of a tilde particle and vice versa. In this way we naturally associate the tilde-particle with a hole. We cannot however, interpret the tilde-particle as a physical anti-particle. For an isolated system, when a quantum of energy is annihilated, we describe it as the action of the operator $\alpha(\mathbf{k})$ on the system. Now, if the system is immersed in a reservoir* , exchange of energy between the system and the reservoir can occur. This temperature effect (exchange of energy) can still be described by the operator $\alpha(\mathbf{k})$ because the effect is manifest only through the structure of the operator $\alpha(\mathbf{k})$ (i.e. a Bogoliubov transformation as explained below). Now, since the presence of the reservoir maintains a certain number of excited quanta, the system can absorb energy in two ways. The first way is through the excitation of additional quanta. In other words, positive energy particles are created in the system. Now, since with each physical particle we associate a hole, the energy of the system can also be raised by annihilating a hole, associated with a particle in the reservoir, in the system. For this second way, we say that an $\tilde{\alpha}$ quantum with negative energy $-\hbar\omega(\mathbf{k})$ and negative momentum $-\hbar\mathbf{k}$ is annihilated in the system.

The operator for the two processes are denoted by $\alpha^{\dagger}(\mathbf{k}, \beta)$ and $\tilde{\alpha}^{\dagger}(\mathbf{k}, \beta)$ respectively (fig.2.1a,b).

---

* The reservoir in field theory has the same role as in ordinary thermodynamics. That is, the temperature of the system is controlled without any changes to the reservoir itself. In QFT, the reservoir is subject to the rules of QFT and we say that it has an infinte number of degrees of freedom.

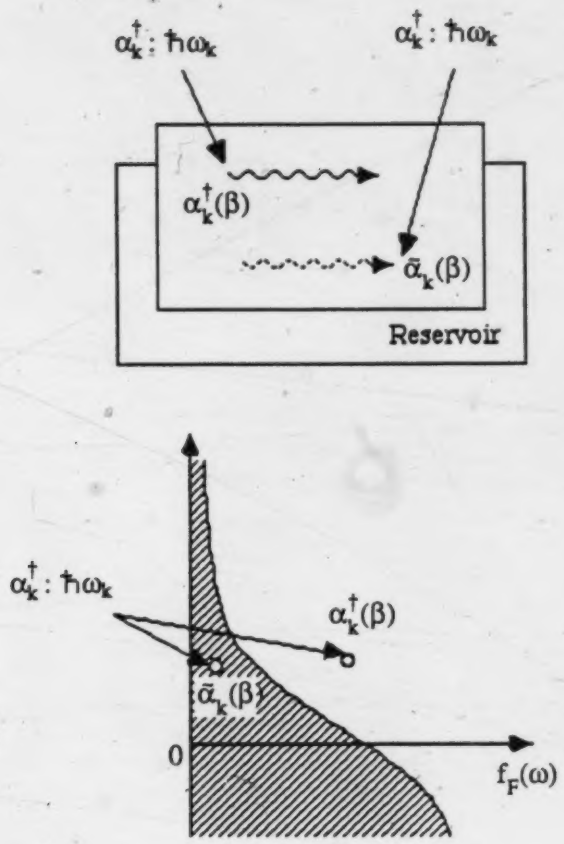


Fig. 2.1 a,b

We thus expect $\alpha(\mathbf{k})$ to be linear combination of $\alpha(\mathbf{k},\beta)$ and $\tilde{\alpha}^\dagger(\mathbf{k},\beta)$. Specifically, we write

$$\alpha(\mathbf{k}) = c(\mathbf{k}, \beta)\alpha(\mathbf{k}, \beta) + d(\mathbf{k}, \beta)\tilde{\alpha}^\dagger(\mathbf{k}, \beta) \qquad (2.2.11)$$

The c-numbers $c(\mathbf{k}, \beta)$ and $d(\mathbf{k}, \beta)$ can be chosen to be real since their phase factor can be absorbed by the annihilation and creation operators. A quantum field theory developed along these lines is equivalent to a thermal Gibbs ensemble. The above interpretation is based upon the work of Kreuzer and Kuper [20] who showed that the system consisting of tilde particles can be interpreted as a representative member of a Gibbs ensemble** .

** The Gibbs ensemble is defined as a large collection of independent replicas of the physical system under study.

Thus by choosing the energy of the tilde particles as negative simply expresses the conservation of energy in the ensemble. As a matter of fact, as Landau and Lifshitz pointed out in their book on statistical mechanics, the statistical mechanics is formulated mostly for open systems. Then the above interpretation is sensible. However, this interpretation needs an external reservoir so that the system is open. Recently, it has been emphasized that the thermo field dynamics can deal with thermal problems in isolated systems. Thus the interpretation of the tilde field requires a deeper consideration. This is being provided by Umezawa in his most recent papers. However, this is out of scope of this review.

Having introduced the tilde-fields, and the thermal vacuum, we now discuss several fundamental relations in TFD, such as the Bogoliubov transformations, the Heisenberg equation, and also the tilde conjugation rules which relate the tilde and the non-tilde particles.

Fundamentals of TFD

The Hamiltonian and Momentum

We have already shown that $\alpha^\dagger(\mathbf{k}, \beta)$ and $\tilde{\alpha}^\dagger(\mathbf{k}, \beta)$ create a quantum of positive energy and momentum and negative energy and momentum respectively, of free observed particles. Hence, the Hamiltonian and momentum for the whole system which includes the reservoir are weakly equal to*

$$\hat{H}_0 = \hbar \int d^3k\, \omega(\mathbf{k}, \beta)\, [\alpha^\dagger(\mathbf{k}, \beta)\alpha(\mathbf{k}, \beta) - \tilde{\alpha}^\dagger(\mathbf{k}, \beta)\tilde{\alpha}(\mathbf{k}, \beta)] \tag{3.1.1}$$

$$\hat{P}_i = \hbar \int d^3k\, k_i [\alpha^\dagger(\mathbf{k}, \beta)\alpha(\mathbf{k}, \beta) - \tilde{\alpha}^\dagger(\mathbf{k}, \beta)\tilde{\alpha}(\mathbf{k}, \beta)] \tag{3.1.2}$$

The Bogoliubov Transformation

When two or more operators satisfy the same commutation relations, they can be related by a Bogoliubov transformation. Since the free particle operators, $\{\alpha(\mathbf{k})\}$, $\{\alpha(\mathbf{k}, \beta)\}$, $\{\tilde{\alpha}(\mathbf{k}, \beta)\}$, associated with free fields satisfy the same commutation relations, (2.2.11) must be a Bogoliubov transformation. Thus we require

$$c^2(\mathbf{k}, \beta) - \sigma d^2(\mathbf{k}, \beta) = 1 \tag{3.2.1}$$

where $\sigma = 1$ for bosons and -1 for fermions. Then we can define

$$\tilde{\alpha}(\mathbf{k}) = c(\mathbf{k}, \beta)\tilde{\alpha}(\mathbf{k}, \beta) + \sigma d(\mathbf{k}, \beta)\alpha^\dagger(\mathbf{k}, \beta) \tag{3.2.2}$$

We can rewrite the above transformation in terms of a certain angle which is determined by the ground state average of the particle number. Thus, we write

---

* An operator is said to be weakly equal to another operator when their expectation values are equal, such as $\langle a| H |b\rangle = \langle a| H_0 |b\rangle + W_0\langle a|b\rangle$. Note that this equality is conditioned by a particular choice of the state vector space.

$$\alpha(k) = \cosh\theta_k\, \alpha(k,\beta) + \sinh\theta_k\, \tilde{\alpha}^{\dagger}(k,\beta) \qquad (3.2.3a)$$

$$\tilde{\alpha}(k) = \cosh\theta_k\, \tilde{\alpha}(k,\beta) + \sinh\theta_k\, \alpha^{\dagger}(k,\beta) \qquad (3.2.3b)$$

for bosons, whereas for fermions we have

$$\alpha(k) = \cos\theta_k\, \alpha(k,\beta) + \sinh\theta_k\, \tilde{\alpha}^{\dagger}(k,\beta) \qquad (3.2.4a)$$

$$\tilde{\alpha}(k) = \cos\theta_k\, \tilde{\alpha}(k,\beta) - \sin\theta_k\, \alpha^{\dagger}(k,\beta) \qquad (3.2.4b)$$

The inverse transformations are

$$\alpha(k,\beta) = \alpha(k)\, \cosh\theta_k - \tilde{\alpha}^{\dagger}(k)\, \sinh\theta_k \qquad (3.2.5a)$$

$$\tilde{\alpha}(k,\beta) = \tilde{\alpha}(k)\, \cosh\theta_k - \alpha^{\dagger}(k)\, \sinh\theta_k \qquad (3.2.5b)$$

for bosons and

$$\alpha(k,\beta) = \alpha(k)\, \cos\theta_k - \tilde{\alpha}^{\dagger}(k)\, \sin\theta_k \qquad (3.2.6a)$$

$$\tilde{\alpha}(k,\beta) = \tilde{\alpha}(k)\, \cos\theta_k + \alpha^{\dagger}(k)\, \sin\theta_k \qquad (3.2.6b)$$

for fermions.

Since, the combination

$$\alpha^{\dagger}(k,\beta)\alpha(k,\beta) - \tilde{\alpha}^{\dagger}(k,\beta)\tilde{\alpha}(k,\beta) \qquad (3.2.7)$$

is invariant under the Bogoliubov transformation, the Hamiltonian and momentum have the form of a free Hamiltonian when expressed in terms of $\alpha(k)$ and $\tilde{\alpha}(k)$. Thus we can write

$$\hat{H}_0 = \hbar \int d^3k\, \omega(k, \beta)\, [\alpha^{\dagger}(k)\alpha(k) - \tilde{\alpha}^{\dagger}(k)\tilde{\alpha}(k)] \qquad (3.2.8)$$

$$\hat{P}_i = \hbar \int d^3k\, k_i [\alpha^{\dagger}(k)\alpha(k) - \tilde{\alpha}^{\dagger}(k)\tilde{\alpha}(k)] \qquad (3.2.9)$$

The operators $\alpha(k, \beta)$ and $\tilde{\alpha}(k, \beta)$ are the annihilation operators at finite temperature which act on the thermal vacuum $|0(\beta)\rangle$ thus

$$\alpha(k, \beta)|0(\beta)\rangle = 0 \qquad (3.2.10a)$$

$$\tilde{\alpha}(k, \beta)|0(\beta)\rangle = 0 \qquad (3.2.10b)$$

The operators $\alpha(k)$, $\tilde{\alpha}(k)$ do not satisfy these relations and therefore cannot be called the annihilation operators.

Determination of θ_k

The angle θ_k is determined by calculating the ground state average of the particle number. Since $\alpha^\dagger(\mathbf{k})$ takes care of the excitations in the system, we can write the number operator of the quanta in the system as

$$N(\mathbf{k}) = \alpha^\dagger(\mathbf{k})\,\alpha(\mathbf{k}) \tag{3.3.1}$$

Hence from the Bogoliubov transformation, the average value of the number density is

$$n(\mathbf{k}) = \frac{(2\pi)^3}{V}\langle 0(\beta)|N(\mathbf{k})|0(\beta)\rangle = \begin{cases} \sinh^2\theta_k & \text{for bosons} \\ \sin^2\theta_k & \text{for fermions} \end{cases} \tag{3.3.2}$$

where* $V = (2\pi)^3\,\delta^{(3)}(0)$ is the volume of the system. When we compare this with the expressions for the number density in statistical mechanics, we find

$$\sinh^2\theta_k = \frac{1}{e^{\beta\omega} - 1} \quad \text{for bosons} \tag{3.3.3a}$$

$$\sin^2\theta_k = \frac{1}{e^{\beta\omega} + 1} \quad \text{for fermions} \tag{3.3.3b}$$

where $\omega = \omega(\mathbf{k}, \beta)$. These relations determine the temperature of the system in a thermal equilibrium state implying that the vacuum expectation value of the number operator is equal to the canonical ensemble average of the particle number. The same holds true for any operator consisting of α and $\alpha^\dagger$.

We will use the following notation

$$f_B(\omega) = \frac{1}{e^{\beta\omega} - 1} \tag{3.3.4a}$$

$$f_F(\omega) = \frac{1}{e^{\beta\omega} + 1} \tag{3.3.4b}$$

So that the Bogoliubov transformation is written as

---

* This is of course true only in the limit sense, when the argument of the delta function, $\mathbf{k}$, approaches zero.

$$\alpha(k) = [1 + f_B(\omega)]^{1/2}\alpha(k, \beta) + \sigma[f_B(\omega)]^{1/2}\tilde{\alpha}^{\dagger}(k, \beta) \qquad (3.3.5a)$$

$$\tilde{\alpha}(k) = [1 + f_B(\omega)]^{1/2}\tilde{\alpha}(k, \beta) + \sigma[f_B(\omega)]^{1/2}\alpha^{\dagger}(k, \beta) \qquad (3.3.5b)$$

for bosons and

$$\alpha(k) = [1 - f_F(\omega)]^{1/2}\alpha(k, \beta) + \sigma[f_F(\omega)]^{1/2}\tilde{\alpha}^{\dagger}(k, \beta) \qquad (3.3.6a)$$

$$\tilde{\alpha}(k) = [1 - f_F(\omega)]^{1/2}\tilde{\alpha}(k, \beta) - \sigma[f_F(\omega)]^{1/2}\alpha^{\dagger}(k, \beta) \qquad (3.3.6b)$$

for fermions. We have the freedom in choosing $\sigma = 1$ or -1. Since $\alpha(k, \beta)$ annihilates the thermal vacuum, the inverse transformations (3.2.5) and (3.2.6) give

$$\alpha(k)|0(\beta)\rangle = \begin{cases} \tanh\theta_k\, \tilde{\alpha}^{\dagger}(k)|0(\beta)\rangle & \text{for bosons} \\ \tan\theta_k\, \tilde{\alpha}^{\dagger}(k)|0(\beta)\rangle & \text{for fermions} \end{cases} \qquad (3.3.7)$$

Notice that there is no sign factor σ in (3.3.7) because we used (3.2.5, 3.2.6). If we expressed for example $\tanh\theta_k$ by (3.3.3a) then we would have a sign factor σ as in (3.3.6a,b). As, it was mentioned before, the $\alpha(k)$ quantum can be interpreted as a hole. This is precisely what equation (3.3.7) tells us; the addition of the $\tilde{\alpha}(k)$ quantum to the vacuum is equivalent to the elimination of the $\alpha(k)$ quantum. Equation (3.3.7) is called the thermal state condition and will be discussed fully later.

The Tilde Operation

Now, we ask how the tilde and the non-tilde operators are related. To answer this question, we begin with the free field(cf. eq. A6.1)*

$$\varphi^0(x) = \sum_r \int d^3k\, [u^r(k, x)\alpha^r(k) + v^r(k, x)\beta^{r\dagger}(k) \qquad (3.4.1)$$

where the index r denotes the spin degree of freedom. Now, the tilde operators specify a quantum with negative energy, $-\omega(k)$, and negative momentum, $-k$. So we define the tilde field to be

---

* The operator $\alpha^r(k)$ for example annihilates a positively charged pion and $\beta^{r\dagger}(k)$ creates a negatively charged pion.

$$\tilde{\varphi}^0(x) = \sum_r \int d^3k \, [u^{r*}(k, x)\tilde{\alpha}^r(k) + v^{r*}(k, x)\tilde{\beta}^{r\dagger}(k) \tag{3.4.2}$$

which satisfies

$$\lambda^*(\partial)\tilde{\varphi}^0(x) = 0 \tag{3.4.3}$$

Now, $u^r(k, x)$ and $v^r(k, x)$ have arbitrary phase factors, so that we can define the following operation

$$\widetilde{c\varphi^0} = c^*\tilde{\varphi}^0 = (\widetilde{c\varphi^{0*}})^* \tag{3.4.4}$$

where c is just a c-number. This is true even when c is a derivative operator. Now from the Bogoliubov transformations we find

$$\tilde{\tilde{\alpha}}(k) = \sigma_F\alpha(k) \quad \tilde{\tilde{\alpha}}(k, \beta) = \sigma_F\alpha(k, \beta) \tag{3.4.5}$$

where $\sigma_F = \begin{cases} +1 & \text{for bosonic operators} \\ -1 & \text{for fermionic operators} \end{cases}$

These rules which are called the tilde conjugation rules are summarized as follows

$$(A_1A_2)^{\sim} = \tilde{A}_1\tilde{A}_2 \tag{3.4.6a}$$

$$(c_1A_1 + c_2A_2)^{\sim} = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2 \tag{3.4.6b}$$

$$\tilde{\tilde{A}} = \rho_F A \tag{3.4.6c}$$

The last result is called the double tilde conjugation and in general is up to the user for its definition. This is because it is a direct consequence of the thermal state condition which depends on a certain phase factor. This phase factor has some freedom in its choice and different choices lead to different double conjugation rules. This will be discussed fully, later.

The Heisenberg Equation

For free fields φ^0, let $H_0[\varphi^0]$ be its Hamiltonian, which is given by

$$H_0[\varphi^0] = \hbar \int d^3k\, \omega(k)\, [\alpha^\dagger(k)\alpha(k) + \beta^\dagger(k)\beta(k)] \qquad (3.5.1)$$

where the spin index has been suppressed. Then, the tilde conjugation rules give

$$\tilde{H}_0 = H_0[\tilde{\varphi}^0]^* = \hbar \int d^3k\, \omega(k)\, [\alpha^\dagger(k)\alpha(k) + \beta^\dagger(k)\beta(k)] \qquad (3.5.2)$$

Now, the free fields as given by (3.4.1) and (3.4.2) give the following results :

$$i\hbar \frac{\partial}{\partial t} \varphi^0(x) = [\varphi^0(x)\,,\, H_0] \qquad (3.5.3)$$

$$i\hbar \frac{\partial}{\partial t} \tilde{\varphi}^0(x) = -[\tilde{\varphi}^0(x)\,,\, \tilde{H}_0] \qquad (3.5.4)$$

These equations are self consistent because the tilde operation on (3.5.3) gives (3.5.4). From equation (3.5.4) we interpret that the Hamiltonian of the tilde field is $-\tilde{H}$. The operators $\beta(k)$ and $\tilde{\beta}(k)$ are related to the physical operators $\beta(k,\beta)$ and $\tilde{\beta}(k,\beta)$ through the Bogoliubov transformation which have the same form as (3.3.5a,b) and (3.3.6a,b).

For interacting Heisenberg fields let $H[\psi]$ be the Hamiltonian consisting of the interacting Heisenberg field ψ. Then

$$\tilde{H} = H^*[\tilde{\psi}^*] \qquad (3.5.6)$$

with the total Hamiltonian given by $\hat{H} = H - \tilde{H}$. The Heisenberg equation for the ψ field is

$$ih \frac{\partial}{\partial t} \psi = [\psi\,,\, H] \qquad (3.5.7)$$

from which we get with the help of the tilde conjugation rules :

$$ih \frac{\partial}{\partial t} \tilde{\psi} = -[\tilde{\psi}\,,\, \tilde{H}] \qquad (3.5.8)$$

Thus, $\hat{H}$ is the total Hamiltonian (i.e. the sum of the usual Hamiltonian and the tilde Hamiltonian). However, the ground state energy is not given by $\langle 0(\beta)|\, \hat{H}\, |0(\beta)\rangle$ but instead by $\langle 0(\beta)|\, H\, |0(\beta)\rangle$. Considering the Lagrangian, if $\mathfrak{L}[\psi]$ is the Lagrangian for ψ then $\tilde{\mathfrak{L}} = \mathfrak{L}^*[\tilde{\psi}^*]$ is the Lagrangian for $\tilde{\psi}$ so that the total Lagrangian is $\hat{\mathfrak{L}} = \mathfrak{L} - \tilde{\mathfrak{L}}$.

The Thermal State Condition

The thermal state condition [21,22] which is also called the tilde-substitution law is the most fundamental relation in TFD. This is because it determines the temperature dependent ground state. From the thermal state condition, it can be shown that Wick's theorem and the Feynman diagram method are available in TFD. Furthermore, the familiar Kubo-Martin-Schwinger (KMS) relation [23] of statistical mechanics is readily derived in TFD from the thermal state condition. We begin by deriving the tilde substitution law.

It will be remembered that the annihilation of the thermal vacuum

$$\alpha(\mathbf{k}, \beta)|0(\beta)\rangle = 0 \qquad (4.1.1a)$$

gave the results

$$\alpha(\mathbf{k}, \beta)|0(\beta)\rangle = \sigma\, e^{-\beta\omega_k/2}\tilde{\alpha}^{\dagger}(\mathbf{k})|0(\beta)\rangle \qquad (4.1.2a)$$

$$\alpha^{\dagger}(\mathbf{k}, \beta)|0(\beta)\rangle = \sigma\rho\, e^{-\beta\omega_k/2}\tilde{\alpha}(\mathbf{k})|0(\beta)\rangle \qquad (4.1.2b)$$

respectively. Equations (4.1.2a,b) are the same as (3.3.7) and are called the thermal state condition. From the thermal state conditions, comes a remarkable result : we can always construct an annihilation operator for the thermal vacuum from any operators A and $\tilde{A}^{\dagger}$. An operator which is constructed in such a way is called a generalized annihilation operator. Thus, we can associate any operator with a generalized annihilation operator. This is very convenient for practical calculations because a wide class of reduction formulas can be derived. In addition, the retarded Green's function can be written in terms of the causal Green's function which is very useful in the response theory.

For writing the thermal state conditions for these generalized annihilation operators, it is required that A can be expanded into an irreducible set of annihilation and creation operators α_i and $\alpha_i^{\dagger}$. For these operators the thermal state condition can be rewritten as

$$\alpha_i(t)|0(\beta)\rangle = \sigma_i\tilde{\alpha}_i^\dagger(t - i\beta/2)|0(\beta)\rangle \tag{4.1.3a}$$

$$\alpha_i^\dagger(t)|0(\beta)\rangle = \sigma'_i\tilde{\alpha}_i(t - i\beta/2)|0(\beta)\rangle \tag{4.1.3b}$$

where $|\sigma_i| = |\sigma'_i| = 1$. Since there is no product of tilde and non-tilde operators in the Heisenberg equations then A can be expanded completely in terms of α_i and $\alpha_i^\dagger$ only while $\tilde{A}$ can be expanded in terms of $\tilde{\alpha}_i$ and $\tilde{\alpha}_i^\dagger$ only. So we write

$$A(t) = \sum_{n,m}\int dt_1\cdots dt_n\, dt'_1\cdots dt'_m\, A_{nm}(t\,;\, t_1,\cdots,t_n\,,\, t'_1,\cdots,t'_m) \times \alpha_{i_1}^\dagger(t_1)\cdots\alpha_{i_n}^\dagger(t_n)\alpha_{j_1}(t'_1)\cdots\alpha_{j_m}(t'_m) \tag{4.1.4a}$$

$$\tilde{A}(t) = \sum_{n,m}\int dt_1\cdots dt_n\, dt'_1\cdots dt'_m\, A_{nm}(t\,;\, t_1,\cdots,t_n\,,\, t'_1,\cdots,t'_m) \times \tilde{\alpha}_{i_1}^\dagger(t_1)\cdots\tilde{\alpha}_{i_n}^\dagger(t_n)\tilde{\alpha}_{j_1}(t'_1)\cdots\tilde{\alpha}_{j_m}(t'_m) \tag{4.1.4b}$$

Now the commutation relations $[\alpha_i\,,\,\alpha_i^\dagger]_{\rho_i} = [\tilde{\alpha}_i\,,\,\tilde{\alpha}_i^\dagger]_{\rho_i} = 1$ and the thermal state conditions (4.1.3) give

$$\sigma'_i = \rho_i\sigma_i^* \tag{4.1.5}$$

where

$$\rho_i = \begin{cases} +1 & \text{for bosonic } \alpha_i \\ -1 & \text{for fermionic } \alpha_i \end{cases}$$

The tilde conjugation rules then give

$$\alpha_i = \rho_i\sigma_i^{*2}\alpha_i \tag{4.1.6a}$$

$$\alpha_i^\dagger = \rho_i\sigma_i^2\alpha_i^\dagger \tag{4.1.6b}$$

Using the thermal state condition, we then get

$$A(t)|0(\beta)\rangle = \sum_{n,m}\int dt_1\cdots dt_n\, dt'_1\cdots dt'_m\, A_{nm}(t - i\beta/2;\, t_1,\cdots,t_n\,,\, t'_1,\cdots,t'_m) \times \sigma_{nm}\tilde{\alpha}_{j_m}^\dagger(t'_m)\cdots\tilde{\alpha}_{j_1}^\dagger(t'_1)\tilde{\alpha}_{i_n}(t_n)\cdots\tilde{\alpha}_{i_1}(t_1)|0(\beta)\rangle \tag{4.1.7}$$

The right hand side of (4.1.7) is, except for the phase factor simply $\tilde{A}^{\dagger}(t - i\beta/2)|0(\beta)\rangle$. The phase factor itself is given by :

$$\sigma_{nm} = (\rho_{i_1}\sigma^{*}_{i_1})\cdots(\rho_{i_n}\sigma^{*}_{i_n})\sigma_{j_1}\sigma_{j_m}(-1)^{F_{nm}(F_{nm}-1)/2} \qquad (4.1.8)$$

where F_{nm} is the number of fermion operators in

$$\alpha^{\dagger}_{i_1}\cdots\alpha^{\dagger}_{i_n}\alpha_{j_1}\cdots\alpha_{j_m}$$

Under certain conditions, the phase factor can be factored out of the summation so that (4.1.7) is just the thermal state condition for the operator A.

The Heisenberg Equation, the Dynamical Map and the Kubo-Martin-Schwinger Equation

The Heisenberg Equation

Since we have two fields in TFD, the tilde and the non-tilde fields, we have a Heisenberg equation for each field as given by

$$\Lambda(\partial)\psi(x) = F[\psi] \tag{5.1.1}$$

$$\Lambda^*(\partial)\tilde{\psi}(x) = F^*[\tilde{\psi}^*] \tag{5.1.2}$$

Both of these equations can be expressed by one equation when we introduce the thermal doublet for any operator F by

$$F^\alpha = \begin{cases} F & \alpha = 1 \\ \tilde{F}^\dagger & \alpha = 2 \end{cases} \tag{5.1.3}$$

If F is a function of other operators, say AB…C so that

$$F = aAB\ldots C$$

for some c-number a. Then F^α is given by

$$F^\alpha = aP_\alpha(A^\alpha B^\alpha \ldots C^\alpha) \tag{5.1.4}$$

where P_α is the ordering operator defined by

$$P_\alpha(A^\alpha B^\alpha \ldots C^\alpha) = \begin{cases} A^1 B^1 \ldots C^1 & \alpha = 1 \\ C^2 \ldots B^2 A^2 & \alpha = 2 \end{cases} \tag{5.1.5}$$

Then in terms of the doublet notation the Heisenberg equation is written as

$$\Lambda(\partial)\psi^\alpha(x) = P_\alpha F[\psi^\alpha(x)] \tag{5.1.6}$$

When $\psi(x)$ is expressed by a column vector then

$$\psi^{\alpha}(x) = \begin{cases} \psi(x) & \alpha = 1 \\ \bar{\psi}^{\dagger}(x)^{t} & \alpha = 2 \end{cases} \tag{5.1.7}$$

where t denotes the transpose operation. When $\psi(x)$ is transformed by a unitary transformation,

$$\psi(x) \rightarrow U\psi(x) \tag{5.1.8}$$

then $\psi^{\alpha}(x)$ is transformed in the same way

$$\psi^{\alpha}(x) \rightarrow U\psi^{\alpha}(x) \tag{5.1.9}$$

The Dynamical Map

Let φ^0 and $\bar{\varphi}^0$ be the free physical fields which satisfy

$$\lambda(\partial)\varphi^0 = 0 \tag{5.2.1}$$

$$\lambda^*(\partial)\bar{\varphi}^0 = 0 \tag{5.2.2}$$

Now we want to write the dynamical map of Heisenberg fields in terms of certain free fields. Since the Heisenberg equations for some ψ and its tilde conjugate (5.1.1,5.1.2) are not coupled, we can choose the free fields φ^0 and $\bar{\varphi}^0$ in such a way that ψ contains only φ^0 while $\bar{\psi}$ contains only $\bar{\varphi}^0$. Then we can write the dynamical map of ψ in the form

$$\psi(x) = \sum_{n,m} \int dk_1 \ldots dk_n dq_1 \ldots dq_m \, F(x; k_1, \ldots, k_n; q_1, \ldots, q_m) \times \alpha^{\dagger}(\mathbf{k}_1) \ldots \alpha^{\dagger}(\mathbf{k}_n) \alpha(\mathbf{q}_1) \ldots \alpha(\mathbf{q}_m) \tag{5.2.3}$$

and the tilde operation gives

$$\bar{\psi}(x) = \sum_{n,m} \int dk_1 \ldots dk_n dq_1 \ldots dq_m \, F^*(x; k_1, \ldots, k_n; q_1, \ldots, q_m) \times \bar{\alpha}^{\dagger}(\mathbf{k}_1) \ldots \bar{\alpha}^{\dagger}(\mathbf{k}_n) \bar{\alpha}(\mathbf{q}_1) \ldots \bar{\alpha}(\mathbf{q}_m) \tag{5.2.4}$$

Here the operators $\alpha(\mathbf{k})$ and $\bar{\alpha}(\mathbf{k})$ are not the physical annihilation operators but are related to $\alpha(\mathbf{k},\beta)$ and $\bar{\alpha}(\mathbf{k},\beta)$ through the Bogoliubov transformation.

The Kubo-Martin-Schwinger Equation

In the statistical mechanics of many body particles, there is the celebrated Kubo-Martin-Schwinger (KMS) relation. In that context, it is derived by employing the identity based on the trace operation which is valid only when finite matrices are involved. It is a tribute to TFD that the KMS relation can be derived in TFD, since in QFT the operators are not associated with finite matrices. In TFD, the thermal state condition is used to derive the KMS relation.

We begin this derivation by considering the thermal state condition as given by (4.1.13a,b) and two operators A(t) and B(t) which are products of Heisenberg fields at time t. Also let n_A be the fermion number of A(t) and n_B that of B(t). Then the fermion number* of $\tilde{A}$ is $-n_A$ and $-n_B$ for $\tilde{B}$. Now, since $\hat{H} = H - \tilde{H}$ is weakly equal to $\hat{H}_0 = H_0 - \tilde{H}_0$ in the calculation of matrix elements of A(t) and B(t), we can use the following expression for the time development of A(t) :

$$A(t) = e^{i\hat{H}_0(t-t')}A(t')e^{-i\hat{H}_0(t-t')} \tag{5.3.1}$$

Now we note the property mentioned earlier for the thermal vacuum :

$$\hat{H}|0(\beta)\rangle = (H - \tilde{H})|0(\beta)\rangle = 0 \tag{5.3.2}$$

With these considerations, we can write the expectation value of $A(t)B(t')$ as

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle = (-1)^{(n_A-1)n_A/2}\langle 0(\beta)|\tilde{A}^{\dagger}(t+i\beta/2)B(t')|0(\beta)\rangle \tag{5.3.3}$$

Since $\tilde{A}$ contains only $\tilde{\varphi}^0$ and B(t) contains only φ^0 then the exchange of $\tilde{A}^{\dagger}(t+i\beta/2)$ and $B(t')$ gives only the sign $(-1)^{n_A n_B}$. Hence we get

---

* Consider the operator $O = \alpha^{\dagger}(k_1)\ldots\alpha^{\dagger}(k_n)\alpha(q_1)\ldots\alpha(q_m)$. The fermion number of O is defined by the difference between the number of creation and annihilation operators in O (or any operator). The operator is bosonic if the fermion number is even and fermionic if it is odd. For a tilde operator, the fermion number is taken to be negative of the corresponding non-tilde operator. This is only to be consistent with the properties of the tilde particle (i.e. negative energy and momentum). The operator remains bosonic or fermionic regardless of the sign of the fermion number.

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle = (-1)^{(n_A-1)n_A/2}(-1)^{n_An_B}\langle 0(\beta)|B(t')\tilde{A}^{\dagger}(t+i\beta/2)|0(\beta)\rangle \quad (5.3.4)$$

Using (4.1.10a) gives us

$$\begin{aligned}&\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle\\ &= (-1)^{(n_A-1)n_A/2}(-1)^{(n_A+1)n_A/2}(-1)^{n_An_B}\langle 0(\beta)|B(t')A(t+i\beta)|0(\beta)\rangle \qquad (5.3.5)\\ &= (-1)^{n_A(n_A+n_B)}\langle 0(\beta)|B(t')A(t+i\beta)|0(\beta)\rangle\end{aligned}$$

Since this vacuum expectation value vanishes for $n_A \neq -n_B$ we get only

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle = \langle 0(\beta)|B(t')A(t+i\beta)|0(\beta)\rangle \quad (5.3.6)$$

for $n_A = n_B$. This is the analogue of the KMS relation at finite temperature and has been referred to as the Kubo-Martin-Schwinger-Takahashi-Umezawa relation [24].

Two-Point Functions in TFD

In the usual quantum field theory, the causal two-point function is defined by

$$G_c(x-y) = \langle 0|T\varphi(x)\varphi^\dagger(y)|0\rangle \tag{6.1.1}$$

However, it is clear that this definition will have to be revised for TFD. Since in TFD there are two types of fields, there are four possible products of fields that can be evaluated. In this sense, the Green's functions in TFD should have a 2 x 2 matrix structure. This is most easily realized when we use the thermal doublet notation :

$$\begin{pmatrix} \varphi^1(x) \\ \varphi^2(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \tilde{\varphi}^\dagger(x) \end{pmatrix} \tag{6.1.2}$$

Using this notation, the causal two-point function in TFD is given by

$$G_c^{\alpha\beta}(x-y) = \langle 0(\beta)|T\varphi^\alpha(x)\varphi^{\beta\dagger}(y)|0(\beta)\rangle \tag{6.1.3}$$

In the same way, we define the retarded and advanced two-point functions :

$$G_r^{\alpha\beta}(x-y) = \theta(t_x - t_y)\langle 0(\beta)|[\varphi^\alpha(x)\,,\,\varphi^{\beta\dagger}(y)]_\pm|0(\beta)\rangle \tag{6.1.4}$$

$$G_a^{\alpha\beta}(x-y) = -\theta(t_x - t_y)\langle 0(\beta)|[\varphi^\alpha(x)\,,\,\varphi^{\beta\dagger}(y)]_\pm|0(\beta)\rangle \tag{6.1.5}$$

where $\theta(x)$ is the step function and the upper sign is for fermion fields while the lower sign is for boson fields. We now discuss examples of fermion and boson systems common in the literature.

Examples of Two Point Functions in TFD

Fermion System;

Here we look at a simple case in which

$$\lambda(\partial) = i\frac{\partial}{\partial t} - \omega(\nabla) \tag{6.2.1}$$

The free fields for this case are given by

$$\varphi^0(x) = (2\pi)^{-3/2}\hbar^{1/2}\int d^3k\,\alpha(k)\,e^{ik\cdot x - i\omega(k)t} \qquad (6.2.2a)$$

$$\tilde{\varphi}^0(x) = (2\pi)^{-3/2}\hbar^{1/2}\int d^3k\,\tilde{\alpha}(k)\,e^{-ik\cdot x + i\omega(k)t} \qquad (6.2.2b)$$

We define the momentum representation of the two-point function from* †

$$S_c^{\alpha\beta}(x-y) = \frac{i\hbar}{(2\pi)^4}\int d^4k\,e^{ik\cdot(x-y) - ik_0(t_x - t_y)}S_c^{\alpha\beta}(k) \qquad (6.2.3)$$

When we calculate $S_c(k)$, we keep in mind that expressions of the type

$$\langle 0(\beta)|\alpha(k)\tilde{\alpha}(k)|0(\beta)\rangle \quad \text{etc;}$$

are calculated from using the Bogoliubov transformation since $\alpha(k)$ etc; are not the annihilation operators of the thermal vacuum. Then letting $\sigma = -1$ in (3.3.6a,b) and using the notation

$$c(\omega) = [1 - f_F(\omega)]^{1/2} \quad , \quad d(\omega) = [f_F(\omega)]^{1/2} \qquad (6.2.4)$$

we easily find $S_c(k)$ as

$$S_c(k) = \begin{pmatrix} \dfrac{c^2(\omega)}{k_0 - \omega + i\delta} + \dfrac{d^2(\omega)}{k_0 - \omega - i\delta} & \dfrac{-c(\omega)d(\omega)}{k_0 - \omega + i\delta} + \dfrac{c(\omega)d(\omega)}{k_0 - \omega - i\delta} \\ \dfrac{-c(\omega)d(\omega)}{k_0 - \omega + i\delta} + \dfrac{c(\omega)d(\omega)}{k_0 - \omega - i\delta} & \dfrac{d^2(\omega)}{k_0 - \omega + i\delta} + \dfrac{c^2(\omega)}{k_0 - \omega - i\delta} \end{pmatrix} \qquad (6.2.5)$$

where δ is a positive infinitesimal and we interpret the fractions as

$$\frac{1}{k_0 - \omega + i\delta} = -\lim_{\delta \to +0}\int_0^\infty dt\, e^{i(k_0 - \omega + i\delta)t} = \frac{\mathcal{P}}{k_0 - \omega} - i\pi\delta(k_0 - \omega)$$

$$\frac{1}{k_0 - \omega - i\delta} = +\lim_{\delta \to +0}\int_{-\infty}^0 dt\, e^{i(k_0 - \omega - i\delta)t} = \frac{\mathcal{P}}{k_0 - \omega} + i\pi\delta(k_0 - \omega)$$

---

* The causal two-point function for fermions is denoted by S_c.

† We have assumed that the system is homogeneous in space and time.

where $\mathcal{P}$ is the denotes the principal value. This two point function can be remarkably simplified into the following form :

$$S_c(k) = U(\omega)\begin{pmatrix} \frac{1}{k_0 - \omega + i\delta} & 0 \\ 0 & \frac{1}{k_0 - \omega - i\delta} \end{pmatrix} U^{\dagger}(\omega)$$

$$= U(\omega)[k_0 - \omega + i\delta\tau]^{-1}\, U^{\dagger}(\omega) \qquad (6.2.6)$$

where $U(\omega)$ is the unitary matrix :

$$U(\omega) = \begin{pmatrix} c(\omega) & d(\omega) \\ -d(\omega) & c(\omega) \end{pmatrix} \qquad (6.2.7)$$

and

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (6.2.8)$$

The retarded and advanced two point functions are calculated in the same way and are found to be

$$S_r^{\alpha\beta}(k) = \frac{\delta^{\alpha\beta}}{k_0 - \omega + i\delta} \qquad (6.2.9)$$

$$S_a^{\alpha\beta}(k) = \frac{\delta^{\alpha\beta}}{k_0 - \omega - i\delta} \qquad (6.2.10)$$

Boson System

Now we consider a boson field system with $\lambda(\partial)$ as given in (6.2.1). The energy $\omega(\mathbf{k})$ in this case must be positive definite. The free fields are again given by (6.2.2a,b). This time we use the Bogoliubov transformation (3.3.5a,b) with $\sigma = +1$. The two point function is given by

$$\begin{aligned}\Delta_c^{\alpha\beta}(x-y) &= \langle 0(\beta)|T[\varphi^{\alpha}(x)\,,\,\varphi^{\beta\dagger}(y)]|0(\beta)\rangle \\ &= \theta(t_x-t_y)\langle 0(\beta)|\varphi^{\alpha}(x)\varphi^{\beta\dagger}(y)]|0(\beta)\rangle \\ &+ \theta(t_y-t_x)\langle 0(\beta)|\varphi^{\beta\dagger}(y)\varphi^{\alpha}(x)]|0(\beta)\rangle\end{aligned} \tag{6.3.1}$$

with the Fourier representation given by

$$\Delta_c^{\alpha\beta}(x-y) = \frac{i\hbar}{(2\pi)^4}\int d^4k\; e^{ik\cdot(x-y)-ik_0(t_x-t_y)}\Delta_c^{\alpha\beta}(k) \tag{6.3.2}$$

After a little work, we get

$$\Delta_c^{\alpha\beta}(k) = \begin{pmatrix} \dfrac{c_B^2(\omega)}{k_0-\omega+i\delta} - \dfrac{d_B^2(\omega)}{k_0-\omega-i\delta} & \dfrac{c_B(\omega)d_B(\omega)}{k_0-\omega+i\delta} - \dfrac{c_B(\omega)d_B(\omega)}{k_0-\omega-i\delta} \\ \dfrac{c_B(\omega)d_B(\omega)}{k_0-\omega+i\delta} - \dfrac{c_B(\omega)d_B(\omega)}{k_0-\omega-i\delta} & \dfrac{d_B^2(\omega)}{k_0-\omega+i\delta} - \dfrac{c_B^2(\omega)}{k_0-\omega-i\delta} \end{pmatrix} \tag{6.3.3}$$

Just as in the previous example, the above result can be greatly simplified to :

$$\Delta_c^{\alpha\beta}(k) = U_B(\omega)\tau[k_0-\omega+i\delta\tau]^{-1}\,U_B^{\dagger}(\omega) \tag{6.3.4}$$

where

$$U_B(\omega) = \begin{pmatrix} c_B(\omega) & d_B(\omega) \\ d_B(\omega) & c_B(\omega) \end{pmatrix} \tag{6.3.5}$$

and

$$c_B(\omega) = [\,1 + f_B(\omega)]^{1/2} \quad , \quad d_B(\omega) = [f_B(\omega)]^{1/2} \tag{6.3.6}$$

We note here that U_B is not unitary but instead satisfies

$$U_B(\omega)\,\tau\,U_B(\omega) = \tau \tag{6.3.7}$$

The retarded and advanced two point functions are defined by

$$\Delta_r^{\alpha\beta}(x-y) = \theta(t_x-t_y)\langle 0(\beta)|[\varphi^{\alpha}(x)\,,\,\varphi^{\beta\dagger}(y)]|0(\beta)\rangle \tag{6.3.8}$$

$$\Delta_r^{\alpha\beta}(x-y) = -\theta(t_x-t_y)\langle 0(\beta)|[\varphi^{\alpha}(x)\,,\,\varphi^{\beta\dagger}(y)]|0(\beta)\rangle \tag{6.3.9}$$

and for this example are found to be

$$\Delta_r^{\alpha\beta}(k) = \frac{\tau^{\alpha\beta}}{k_0 - \omega + i\delta} \tag{6.3.10}$$

$$\Delta_a^{\alpha\beta}(k) = \frac{\tau^{\alpha\beta}}{k_0 - \omega - i\delta} \tag{6.3.11}$$

The Klein-Gordon Field

We now consider a slightly more complicated case in which

$$\lambda(\partial) = -\left[\frac{\partial}{\partial t^2} + \omega^2(\nabla)\right] \tag{6.4.1}$$

In this case $\rho = 1$ so that φ^0 must be a boson field. Also, since any complex field is a linear combination of real fields we can without any loss of generality assume that φ^0 is a real field (i.e. $\varphi^{0\dagger} = \varphi^0$). The fields are then given by

$$\varphi^0(x) = (2\pi)^{-3/2}\hbar^{1/2}\int \frac{d^3k}{(2\omega)^{1/2}}\{\alpha(k)\, e^{i(k\cdot x - \omega t)} + \alpha^\dagger(k)\, e^{-i(k\cdot x - \omega t)}\} \tag{6.4.2a}$$

$$\tilde{\varphi}^0(x) = (2\pi)^{-3/2}\hbar^{1/2}\int \frac{d^3k}{(2\omega)^{1/2}}\{\tilde{\alpha}(k)\, e^{-i(k\cdot x - \omega t)} + \tilde{\alpha}^\dagger(k)\, e^{i(k\cdot x - \omega t)}\} \tag{6.4.2b}$$

Again we use the Bogoliubov transformation (3.35a,b) with $\sigma = 1$. Then the two point functions are calculated to be

$$\Delta_c(k) = U_B(\omega)\,\tau\,[k_0^2 - (\omega - i\delta\tau)^2\,]^{-1}U_B(\omega) \tag{6.4.3}$$

$$\Delta_r^{\alpha\beta}(k) = \frac{\tau^{\alpha\beta}}{(k_0 + i\delta)^2 - \omega^2} \tag{6.4.4}$$

$$\Delta_a^{\alpha\beta}(k) = \frac{\tau^{\alpha\beta}}{(k_0 - i\delta)^2 - \omega^2} \tag{6.4.5}$$

Spectral Representation of Two-Point Functions

We will now derive the spectral representation for a two point function of operators A(x) and B(y). The spectral representation is very useful for many general analyses and also for the analyses of response functions in linear response theory. We begin by letting n_A be the fermion number of the A(x) and n_B that of B(y). We also use the thermal doublet notation introduced earlier :

$$\begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \equiv \begin{pmatrix} A \\ \tilde{A}^\dagger \end{pmatrix} \qquad \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \equiv \begin{pmatrix} B \\ \tilde{B}^\dagger \end{pmatrix} \tag{7.1.1}$$

Now we write the causal two-point function as

$$\begin{aligned} G^{\alpha\beta}_{cAB}(x-y) &= \langle 0(\beta)|T\, A^\alpha(x)B^\beta(y)|0(\beta)\rangle \\ &= \theta(t_x - t_y)\langle 0(\beta)|A^\alpha(x)B^\beta(y)|0(\beta)\rangle \\ &+ \rho\theta(t_y - t_x)\langle 0(\beta)|B^\beta(y)A^\alpha(x)|0(\beta)\rangle \end{aligned} \tag{7.1.2}$$

where $\rho = 1$ for boson like A and B and -1 for fermion like A,B. Now we insert into this definition the following expression for the step function

$$\theta(t) = \lim_{\delta \to +0} \frac{i}{2\pi} \int dw \frac{1}{w + i\delta} e^{-iwt} \tag{7.1.3}$$

and we also insert between the two operators A(x) and B(y) the complete set :

$$\sum_{n,m} |n,\tilde{m}\rangle\langle\tilde{m},n|$$

so that we get the following Fourier amplitude of (7.1.2) :

$$G^{\alpha\beta}_{cAB}(k) = \int_{-\infty}^{\infty} dw \left[\frac{\sigma^{\alpha\beta}_{AB}(w,\mathbf{k})}{k_0 - w + i\delta} - \rho \frac{\sigma^{\alpha\beta}_{BA}(w,-\mathbf{k})}{k_0 + w - i\delta} \right] \tag{7.1.4}$$

where

$$\sigma^{\alpha\beta}_{AB}(w,k) = (2\pi)^3 \sum_{n,m} \langle 0(\beta)|A^{\alpha}(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B^{\beta}(0)|0(\beta)\rangle \times \delta(p - p_n + p_m)\delta(w - E_n + E_m) \tag{7.1.5}$$

In this result, p_n and E_n are the momentum and energy of the n-state while p_m and E_m are the momentum and energy of the m-state.

Before we continue further, we note the following important result required for the derivation of the spectral representation. Let $|n,\tilde{m}\rangle$ be a state vector constructed from the n-times cyclic operation of $\alpha^{\dagger}(\beta)$ (i.e. $(\alpha^{\dagger}(\beta))^n|0(\beta)\rangle$) and m times cyclic operation of $\tilde{\alpha}^{\dagger}(\beta)$ on $|0(\beta)\rangle$. Then the tilde operation gives

$$\begin{aligned}\langle n,\tilde{m}|\psi(x)|\tilde{n}',\tilde{m}'\rangle &= \langle\tilde{n},\tilde{m}|\tilde{\psi}(x)|\tilde{n}',\tilde{m}\rangle^{*} \\ &= (-1)^{n_m - n_{m'}}\langle\tilde{n},m|\tilde{\psi}(x)|\tilde{n}',m'\rangle\end{aligned} \tag{7.1.6}$$

where n_m is the number of fermions in the $|\tilde{m}\rangle$ state. Using this and the thermal state condition, we get the following useful relations :

$$\langle 0(\beta)|\tilde{A}^{\dagger}(0)|n,\tilde{m}\rangle = (-1)^{F_A(F_A-1)/2} e^{-\beta(E_n - E_m)/2}\langle 0(\beta)|A(0)|n,\tilde{m}\rangle \tag{7.1.7a}$$

$$\begin{aligned}\langle n,\tilde{m}|A(0)|0(\beta)\rangle &= (-1)^{n_m}\langle\tilde{n},m|\tilde{A}(0)|0(\beta)\rangle^{*} \\ &= (-1)^{n_m}\langle 0(\beta)|\tilde{A}^{\dagger}(0)|m,\tilde{n}\rangle \\ &= (-1)^{n_m}(-1)^{F_A(F_A-1)/2} e^{-\beta(E_m - E_n)/2}\langle 0(\beta)|A(0)|m,\tilde{n}\rangle\end{aligned} \tag{7.1.7b}$$

$$\begin{aligned}\langle n,\tilde{m}|\tilde{A}^{\dagger}(0)|0(\beta)\rangle &= (-1)^{F_A(F_A+1)/2} e^{-\beta(E_n - E_m)/2}\langle n,\tilde{m}|A(0)|0(\beta)\rangle \\ &= (-1)^{F_A}(-1)^{n_m}\langle 0(\beta)|A(0)|m,\tilde{n}\rangle\end{aligned} \tag{7.1.7c}$$

$$\langle n,\tilde{m}|\tilde{B}^{\dagger}(0)|0(\beta)\rangle = (-1)^{F_B(F_B+1)/2} e^{-\beta(E_n - E_m)/2}\langle n,\tilde{m}|B(0)|0(\beta)\rangle \tag{7.1.8a}$$

$$\langle 0(\beta)|B(0)|n,\tilde{m}\rangle = (-1)^{n_m}(-1)^{F_B(F_B+1)/2} e^{-\beta(E_m - E_n)/2}\langle m,\tilde{n}|B(0)|0(\beta)\rangle \tag{7.1.8b}$$

$$\langle 0(\beta)|\tilde{B}^{\dagger}(0)|n,\tilde{m}\rangle = (-1)^{F_B}(-1)^{n_m}\langle m,n|B(0)|0(\beta)\rangle \tag{7.1.8c}$$

From these relations we get

$$\langle 0(\beta)|A^{\alpha}(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B^{\beta}(0)|0(\beta)\rangle = \langle 0(\beta)|A(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B(0)|0(\beta)\rangle \times \begin{pmatrix} 1 & (-1)^{F_A(F_A-1)/2} e^{-\beta(E_n-E_m)/2} \\ (-1)^{F_A(F_A-1)/2} e^{-\beta(E_n-E_m)/2} & e^{-\beta(E_n-E_m)/2} \end{pmatrix}^{\alpha\beta} \tag{7.1.9}$$

$$\langle 0(\beta)|B^{\beta}(0)|m,\tilde{n}\rangle\langle\tilde{n},m|A^{\alpha}(0)|0(\beta)\rangle = \langle 0(\beta)|A(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B(0)|0(\beta)\rangle \times \begin{pmatrix} e^{-\beta(E_n-E_m)/2} & (-1)^{F_A(F_A+1)/2} e^{-\beta(E_n-E_m)/2} \\ (-1)^{F_A(F_A+1)/2} e^{-\beta(E_n-E_m)/2} & 1 \end{pmatrix}^{\alpha\beta} \tag{7.1.10}$$

when $F_A = -F_B$. When $F_A \neq -F_B$, then $G^{\alpha\beta}_{cAB} = 0$. Thus in the rest of this discussion we assume that $F_A = -F_B$.

With above results let us write (7.1.5) as

$$\sigma^{\alpha\beta}_{AB}(w,\mathbf{p}) = \sigma_{AB}(w,\mathbf{p}) \begin{pmatrix} c^2(w) & (-1)^{F_A(F_A-1)/2} c(w)d(w) \\ (-1)^{F_A(F_A+1)/2} c(w)d(w) & d^2(w) \end{pmatrix}^{\alpha\beta} \tag{7.1.11a}$$

$$\sigma^{\alpha\beta}_{AB}(-w,-\mathbf{p}) = \sigma_{AB}(w,\mathbf{p}) \begin{pmatrix} d^2(w) & (-1)^{F_A(F_A+1)/2} c(w)d(w) \\ (-1)^{F_A(F_A+1)/2} c(w)d(w) & c^2(w) \end{pmatrix}^{\alpha\beta} \tag{7.1.11b}$$

with

$$\sigma_{AB}(w,\mathbf{p}) = \frac{(2\pi)^3}{\hbar}(1-\rho\, e^{-\beta\omega}) \sum_{n,m} \delta(\mathbf{p}-\mathbf{p}_n+\mathbf{p}_m)\delta(w-E_n+E_m) \times \langle 0(\beta)|A(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B(0)|0(\beta)\rangle \tag{7.1.12}$$

also

$$c^2(w) = 1 + \rho\, f(w) \qquad d^2(w) = f(w) \tag{7.1.13}$$

$$f(w) = \frac{1}{e^{\beta\omega} - \rho} \tag{7.1.14}$$

When we compare (7.11a,b) with (6.2.6) and (6.3.4) we find the following results :

$$G^{\alpha\beta}_{cAB}(k) = \int_{-\infty}^{\infty} dw\, \sigma_{AB}(w,k) \times (\tau^{(F_A+1)/2} U(w)[k_0 - w + i\delta\tau]^{-1} U^{\dagger}(w) \tau^{(F_A+1)/2})^{\alpha\beta} \tag{7.1.15}$$

for fermions while

$$G^{\alpha\beta}_{cAB}(k) = \int_{-\infty}^{\infty} dw\, \sigma_{AB}(w,k)\, (\tau^{F_A/2} U_B(w) \tau [k_0 - w + i\delta\tau]^{-1} U_B(w) \tau^{F_A/2})^{\alpha\beta} \tag{7.1.16}$$

for bosons. Equations (7.1.15) and (7.1.16) are the spectral representation of the causal two point functions at finite temperature.

We can also derive the spectral representation of the retarded and advanced two point functions. We write them as

$$G^{\alpha\beta}_{rAB}(x - y) = \theta(t_x - t_y) \langle 0(\beta) | [A^{\alpha}(x)\, ,\, B^{\beta}(y)]_{\pm} | 0(\beta) \rangle \tag{7.1.17}$$

$$G^{\alpha\beta}_{aAB}(x - y) = -\theta(t_y - t_x) \langle 0(\beta) | [A^{\alpha}(x)\, ,\, B^{\beta}(y)]_{\pm} | 0(\beta) \rangle \tag{7.1.18}$$

where the upper sign is for fermion like A,B and the lower for boson like A,B. Then employing the same method as for the causal two point functions, we find the following spectral representation for the retarded and advanced two point functions :

$$G^{\alpha\beta}_{rAB}(k) = \delta^{\alpha\beta}\int_{-\infty}^{\infty} dw\, \frac{\sigma_{AB}(w, \mathbf{k})}{k_0 - w + i\delta} \tag{7.1.19}$$

$$G^{\alpha\beta}_{aAB}(k) = \delta^{\alpha\beta}\int_{-\infty}^{\infty} dw\, \frac{\sigma_{AB}(w, \mathbf{k})}{k_0 - w - i\delta} \tag{7.1.20}$$

for fermion like operators and

$$G^{\alpha\beta}_{rAB}(k) = \tau^{\alpha\beta}\int_{-\infty}^{\infty} dw\, \frac{\sigma_{AB}(w, \mathbf{k})}{k_0 - w + i\delta} \tag{7.1.21}$$

$$G^{\alpha\beta}_{aAB}(k) = \tau^{\alpha\beta}\int_{-\infty}^{\infty} dw\, \frac{\sigma_{AB}(w, \mathbf{k})}{k_0 - w - i\delta} \tag{7.1.22}$$

for boson like operators.

We can derive a sum rule for the spectral function σ_{AB}. For this we note that from (7.1.18) that

$$\langle 0(\beta)|A^{\alpha}(x)^{\beta}(y)|0(\beta)\rangle = \frac{\hbar}{(2\pi)^3}\int d^3p \int dw,\, e^{ik\cdot(x-y) - iw(t_x - t_y)}\, \sigma_{AB}(w,\mathbf{p}) \times \begin{pmatrix} c^2(w) & \cdots & (-1)^{F_A(F_A-1)/2} c(w)d(w) \\ (-1)^{F_A(F_A-1)/2} c(w)d(w) & & d^2(w) \end{pmatrix} \tag{7.1.23}$$

and

$$\langle 0(\beta)|[A^{\alpha}(x)\,,\, B^{\beta}(y)]|0(\beta)\rangle_{t_x = t_y} = \frac{\hbar}{(2\pi)^3}\int d^3k\; e^{ik\cdot(x-y)}\int dw\; \sigma_{AB}(w,\,\mathbf{p})(\tau^{(1+\rho)/2})^{\alpha\beta} \tag{7.1.24}$$

Then if B is the canonical conjugate of A i.e if

$$[A^{\alpha}(x)\,,\,B^{\beta}(y)]_{\pm} = h(\tau^{(1+\rho)/2})^{\alpha\beta}\delta^{(4)}(x-y) \qquad (7.1.25)$$

then with the help of (7.1.24) we get the following sum rule

$$\int_{-\infty}^{\infty} dw\ \sigma_{AB}(w,\mathbf{p}) = 1 \qquad (7.1.26)$$

Single Particle States

When there are single particle states with energy $\omega(\mathbf{k})$ participating in the summation of (7.1.12) then $\sigma_{AB}(w,\mathbf{k})$ has the following form

$$\sigma_{AB}(w,\mathbf{k}) = Z_{AB}(\mathbf{k})\delta[w-\omega(\mathbf{k})] + \tilde{\sigma}_{AB}(w,\mathbf{k}) \qquad (7.2.1)$$

At finite temperature $\sigma_{AB}(w,\mathbf{k})$ can assume non-vanishing values for both positive and negative w because the sign of $E_n - E_m$ is not arbitrary. Now from (7.1.7b,7.1.8b) we get

$$\langle 0(\beta)|A(0)|n,\tilde{m}\rangle\langle\tilde{m},n|B(0)|0(\beta)\rangle = e^{-\beta(E_n-E_m)}\langle 0(\beta)|B(0)|\tilde{m},m\rangle\langle m,\tilde{m}|A(0)|0(\beta)\rangle \qquad (7.2.2)$$

so that

$$\sigma_{BA}(w,\mathbf{p}) = -\rho\sigma_{BA}(-w,\mathbf{p}) \qquad (7.2.3)$$

If the theory is invariant under spatial reflection then

$$\sigma_{AB}(w,\mathbf{p}) = -\rho\sigma_{BA}(-w,-\mathbf{p}) \qquad (7.2.4)$$

Now when A is a boson like operator and B=A then the spectral representation of the causal two-point function is

$$G^{\alpha\beta}_{cAA}(k) = \int dw\ \tilde{\sigma}_{AA}(w,k) \times (\tau^{F\sqrt{2}}U_B(w)\tau[k_0^2 - (w - i\delta\tau)^2]^{-1}U_B(w)\tau^{F\sqrt{2}})^{\alpha\beta} \qquad (7.2.5)$$

where

$$\tilde{\sigma}_{AA}(w,\mathbf{k}) = 2w\,\sigma_{AA}(w,\mathbf{k}) \qquad (7.2.6)$$

Similarly the spectral representation of the retarded and advanced two-point functions are

$$G^{\alpha\beta}_{rAA}(k) = \tau^{\alpha\beta}\int_0^\infty dw\, \tilde{\sigma}_{AA}(w, \mathbf{k})\frac{1}{(k_0 + i\delta)^2 - w^2} \tag{7.2.7}$$

$$G^{\alpha\beta}_{aAA}(k) = \tau^{\alpha\beta}\int_0^\infty dw\, \tilde{\sigma}_{AA}(w, \mathbf{k})\frac{1}{(k_0 - i\delta)^2 - w^2} \tag{7.2.8}$$

Analytic Properties of the Causal Green's Function

It was mentioned earlier that the 1-1 component of the causal Green's function corresponds to the conventional Green function in zero temperature QFT. Thus for the case of fermions this is given by

$$S_c^{11}(p) = \int dw\, \sigma(w, \mathbf{p}) \left[\frac{c^2(w)}{p_0 - w + i\delta} + \frac{d^2(w)}{p_0 - w - i\delta} \right] \tag{8.1.1}$$

$$S_r^{11}(p) = \int dw\, \sigma(w, \mathbf{p}) \frac{1}{p_0 - w + i\delta} \tag{8.1.2}$$

$$S_a^{11}(p) = \int dw\, \sigma(w, \mathbf{p}) \frac{1}{p_0 - w - i\delta} \tag{8.1.3}$$

For analyzing the analytic properties of these functions, we consider the following function defined in the complex z plane

$$S(z, \mathbf{p}) = \int dw\, \sigma(w, \mathbf{p}) \frac{1}{z - w} \tag{8.1.4}$$

This function is singular along the real z-axis but is analytic in the upper and lower half planes of z. Thus we say that the retarded and advanced Green's functions are the boundary values of S(z,**p**). This is made clearer with the following dispersion relation

$$\mathrm{Re}\, G_c(z, \mathbf{p}) = -\frac{1}{\pi} \int dw\, \frac{\mathcal{P}}{z - w} \frac{e^{\beta w} + 1}{e^{\beta w} - 1} \mathrm{Im}\, G_c(z, \mathbf{p}) \tag{8.1.5}$$

Here, $\mathcal{P}$ denotes the principal value of the integral. Even though the causal Green's function is not analytic on the real z-axis, it is regarded as an analytic function with cuts along $x \pm i\delta$. This can be seen by considering the following functions :

$$G_I^+(p_0) = -i\int dt\, e^{ip_0t}\,\theta(t)\langle 0(\beta)|\psi(t)\psi^\dagger(0)|0(\beta)\rangle \tag{8.1.6}$$

$$G_I^-(p_0) = \;\; i\int dt\, e^{ip_0t}\,\theta(-t)\langle 0(\beta)|\psi(t)\psi^\dagger(0)|0(\beta)\rangle \tag{8.1.7}$$

$$G_{II}^+(p_0) = -i\int dt\, e^{ip_0t}\,\theta(t)\langle 0(\beta)|\psi^\dagger(0)\psi(t)|0(\beta)\rangle \tag{8.1.8}$$

$$G_{II}^-(p_0) = \;\; i\int dt\, e^{ip_0t}\,\theta(-t)\langle 0(\beta)|\psi^\dagger(0)\psi(t)|0(\beta)\rangle \tag{8.1.9}$$

where the trivial dependence of the space variables has been omitted. The spectral representation of these functions is given by

$$G_I^+(p_0) = \int dw\, \sigma(w)\, \frac{1}{p_0 - w + i\varepsilon}\, \frac{e^{\beta w}}{e^{\beta w}+1} \tag{8.1.10}$$

$$G_I^-(p_0) = \int dw\, \sigma(w)\, \frac{1}{p_0 - w - i\varepsilon}\, \frac{e^{\beta w}}{e^{\beta w}+1} \tag{8.1.11}$$

$$G_{II}^+(p_0) = \int dw\, \sigma(w)\, \frac{1}{p_0 - w + i\varepsilon}\, \frac{1}{e^{\beta w}+1} \tag{8.1.12}$$

$$G_{II}^-(p_0) = \int dw\, \sigma(w)\, \frac{1}{p_0 - w - i\varepsilon}\, \frac{1}{e^{\beta w}+1} \tag{8.1.13}$$

It is clear that $G_I^+(p_0)$ and $G_{II}^+(p_0)$ are analytic in the upper half of the complex p_0 plane while $G_I^-(p_0)$ and $G_{II}^-(p_0)$ are analytic in the lower half plane. Thus, in analogy to (8.1.4) we introduce the functions

$$G_I(z) = \int dw\, \sigma(w)\, \frac{1}{z - w}\, \frac{e^{\beta w}}{e^{\beta w}+1} \tag{8.1.14}$$

$$G_{II}(z) = \int dw\, \sigma(w)\, \frac{1}{z - w}\, \frac{1}{e^{\beta w}+1} \tag{8.1.15}$$

Now although one might think that the analytic continuation of $G_I(z)$ from the upper half plane leads to $G_I^-(z)$, this is in fact not the case. Since there is a cut along the real axis, we must transfer to a second Riemann sheet. Thus the analytic continuation of $G_I^+(z)$ into the

lower half plane gives $G_I^{+'}(z)$ instead of $G_I^-(z)$. The same consideration applies to the other functions as illustrated in fig (8.1a,b)

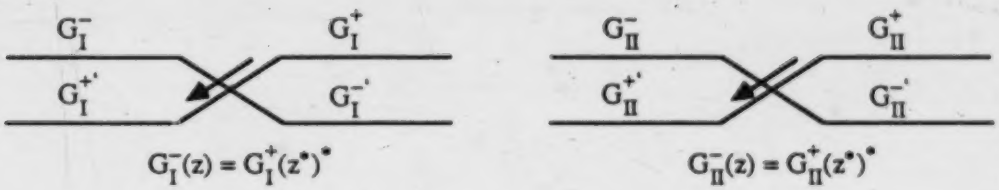


Fig. 8.1

We now define the function

$$G_c(z,\delta) = G_I(z + i\delta) + G_{II}(z - i\delta) \qquad (8.1.16)$$

which is clearly analytic on the real axis. However since it has two poles, the z-plane is divided into three regions by three Riemann sheets as shown in fig.(8.2 a,b) the upper sheet is called the physical sheet, so that $G_c(z,\delta)$ is the analytic function with cuts along $x \pm i\delta$.

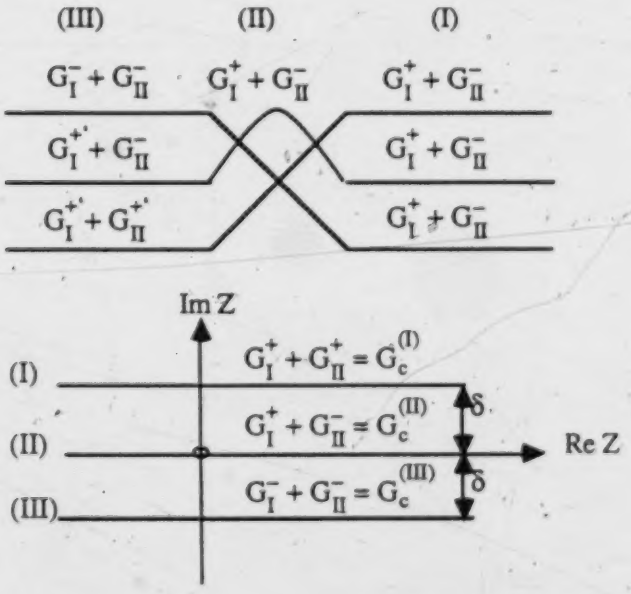


Fig. 8.2

Then $G_c(z,\delta)$ in the three regions is labelled $G_c^{I}(z,\delta)$, $G_c^{II}(z,\delta)$, $G_c^{III}(z,\delta)$ respectively. Hence the causal Green's function is obtained from $G_c^{II}(z,\delta)$ in the limit $\delta\to 0$. The function in each of the three regions in this limit is given by

$$G_c^{I}(z) = \mathcal{P}\int dw\,\frac{\sigma(w)}{x-w} - i\pi\sigma(x) \tag{8.1.17}$$

$$G_c^{II}(z) = \mathcal{P}\int dw\,\frac{\sigma(w)}{x-w} - i\pi\,\frac{e^{\beta x}-1}{e^{\beta x}+1}\,\sigma(x) \tag{8.1.18}$$

$$G_c^{III}(z) = \mathcal{P}\int dw\,\frac{\sigma(w)}{x-w} - i\pi\sigma(x) \tag{8.1.19}$$

Thus we have obtained the analytic continuation of the causal Green's function onto the real axis. Also the retarded and advanced Green's functions are given by $G_c^{I}(z)$ and $G_c^{III}(z)$

Bethe-Salpeter Equations at Finite Temperature

The Bethe-Salpeter equations [25] in the usual QFT are useful for bound state problems* , even though one usually has to resort to various approximational methods because of the difficulty of solving these equations. We now derive the analog equations in TFD. Consider then a set of Heisenberg fields ψ_a satisfying the Heisenberg equation

$$\lambda_{ab}(\partial)\psi_b = F_b[\psi] \tag{9.1.1}$$

which we can write as

$$\lambda(\partial)\psi = F[\psi] \tag{9.1.2}$$

where ψ and $F[\psi]$ are column vectors with components ψ_a and F_a respectively. Then from the Heisenberg equation expressed in the doublet notation we get

$$\lambda(\partial)\psi^\alpha = P_\alpha\, F[\psi^\alpha] \tag{9.1.3}$$

From the two point function

$$\begin{aligned} G^{\alpha\beta}(x,y) &= \langle 0(\beta)|T\,[\psi^\alpha(x)\,,\,\psi^{\beta\dagger}(y)]|0(\beta)\rangle \\ &= \theta(t_x - t_y)\langle 0(\beta)|\psi^\alpha(x)\psi^{\beta\dagger}(y)|0(\beta)\rangle \\ &\mp \theta(t_y - t_x)\langle 0(\beta)|\psi^{\beta\dagger}(y)\psi^\alpha(x)|0(\beta)\rangle \end{aligned} \tag{9.1.4}$$

we calculate the operation of $\lambda(\partial)$ on $G^{\alpha\beta}$ by using (3) to obtain

$$\lambda(\partial)G^{\alpha\beta}(x,y) = F^{\alpha\beta}(x,y) + i\hbar\delta^{(4)}(x-y)\delta^{\alpha\beta} \tag{9.1.5}$$

for fermions and

---

* In QFT, the bound states and resonances of a system are identified by the occurrence of poles in the Green's functions. Since the Bethe-Salpeter equations are integral equations for the Green function, they can be very useful for bound state problems.

$$\lambda(\partial)G^{\alpha\beta}(x,y) = \{\tau F(x,y)\}^{\alpha\beta} + i\hbar\delta^{(4)}(x-y)\delta^{\alpha\beta} \qquad (9.1.6)$$

for bosons.

Here $F^{\alpha\beta}(x,y)$ is some function. We can put these into the following integral form

$$G^{\alpha\beta}(x,y) = S^{\alpha\beta}(x-y) - \frac{i}{\hbar}\int d^4z\, S^{\alpha\beta}(x-z)\, F^{\alpha\beta}(z,y) \qquad (9.1.7)$$

for fermions and

$$G^{\alpha\beta}(x,y) = \Delta^{\alpha\beta}(x-y) - \frac{i}{\hbar}\int d^4z\, \Delta^{\alpha\beta}(x-z)\, F^{\alpha\beta}(z,y) \qquad (9.1.8)$$

for bosons.

In deriving these results we note that when F in (2) vanishes then G becomes S or Δ. Also the function $F^{\rho\beta}$ may contain multi-point functions

$$<0(\beta)|T[\psi^{\alpha}(x)\psi^{\beta}(y)\psi^{\gamma}(z)\ldots]|0(\beta)>$$

Integral equations for these multi-point functions can also be derived. These integral equations are the Bethe-Salpeter equations.

The Feynman Propagators

In non-relativistic scattering problems, one looks for solutions which develop in time from initial conditions imposed in the remote past. In other words, given a wave packet which in the remote past represents a particle approaching a potential, one asks what the wave will look like in the remote future. The solution can be expressed generally as

$$\psi(\mathbf{x}',t') = i\int d^3x\, G(\mathbf{x}',t';\mathbf{x},t)\psi(\mathbf{x},t) \qquad t' > t \tag{10.1.1}$$

The function $G(\mathbf{x}',t';\mathbf{x},t)$ is known as the Green's function or propagator. The propagator can also be expressed as the solution of a certain differential equation. This idea can be generalized to relativistic QFT, where the propagator also satisfies a differential equation.

For the theory of quantum electrodynamics, which is a theory for the interaction of radiation (photon) and matter (electron) fields the propagators are given by:

$$\Delta_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\delta} \tag{10.1.2}$$

for the Klein-Gordon field (photon) and

$$S(k) = \frac{i(\not{k} + m)}{k^2 - m^2 + i\delta} \tag{10.1.3}$$

for the Dirac field (electron). So we begin by calculating the photon and electron propagators in TFD.

The Photon Propagator

The Klein-Gordon field satisfies the equation

$$-\left[\frac{\partial^2}{\partial t^2}+\omega^2(\nabla)\right]\varphi^0(x)=0 \tag{10.2.1}$$

where $\omega^2(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}=\omega^2(\nabla)e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\omega^2(\mathbf{k})=\mathbf{k}^2$. The free field satisfying (10.2.1) is given by

$$\varphi^0(x)=(2\pi)^{-3/2}h\int\frac{d^3k}{(2\omega)^{1/2}}\left[\alpha(\mathbf{k})\exp\{i(\mathbf{k}\cdot\mathbf{x}-\omega t)\}+\alpha^\dagger(\mathbf{k})\exp(-i(\mathbf{k}\cdot\mathbf{x}-\omega t))\right] \tag{10.2.2}$$

so that

$$\tilde{\varphi}^0(x)=(2\pi)^{-3/2}h\int\frac{d^3k}{(2\omega)^{1/2}}\left[\tilde{\alpha}(\mathbf{k})\exp\{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)\}+\tilde{\alpha}^\dagger(\mathbf{k})\exp(i(\mathbf{k}\cdot\mathbf{x}-\omega t))\right] \tag{10.2.3}$$

The expression (10.2.2) is obtained under the requirement that $\varphi^0(x)$ is real (i.e. $\varphi^{0\dagger}(x)=\varphi^0(x)$).

The two point function is then readily calculated using :

$$\begin{aligned}\Delta_c^{\alpha\beta}(x-y)&=\langle 0(\beta)|T\,\varphi^\alpha(x)\varphi^{\beta\dagger}(y)|0(\beta)\rangle\\&=\theta(t_x-t_y)\langle 0(\beta)|\varphi^\alpha(x)\varphi^{\beta\dagger}(y)|0(\beta)\rangle\\&\quad+\theta(t_y-t_x)\langle 0(\beta)|\varphi^{\beta\dagger}(y)\varphi^\alpha(x)|0(\beta)\rangle\end{aligned} \tag{10.2.4}$$

with the Fourier representation given by

$$\Delta_c^{\alpha\beta}(x-y)=\frac{1}{h}\int d^4k\,e^{-ik\cdot(x-y)}\Delta_c^{\alpha\beta}(k) \tag{10.2.5}$$

Thus as discussed earlier, the result is :

$$-i\Delta_c(k) = U_B(\omega)\tau[k_0^2 - (\omega - i\delta\tau)^2]^{-1}U_B(\omega) \qquad (10.2.6)$$

where $U_B(\omega)$ is given by (6.3.5).

The electron propagator

The Dirac field satisfies

$$(i\partial - m)\psi^0(x) = 0 \qquad (10.3.1)$$

where $\partial = \gamma^\mu\partial_\mu$. When we write

$$\lambda(\partial) = i\partial - m \qquad (10.3.2)$$

we find that the divisor for the Dirac equation is

$$d(\partial) = i\partial + m \qquad (10.3.3)$$

so that the Dirac equation is a type two equation because

$$\lambda(\partial)d(\partial) = -\left[\frac{\partial^2}{\partial t^2} + \varepsilon^2(\nabla)\right] \qquad (10.3.4)$$

where

$$\varepsilon^2(\nabla) = m^2 - \nabla^2 \qquad (10.3.5)$$

Here and in the following, by $\varepsilon^2(\nabla)$ we mean

$$\varepsilon^2(\nabla)e^{i\mathbf{k}\cdot\mathbf{x}} = \varepsilon^2(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \qquad (10.3.6)$$

with $\varepsilon^2(\mathbf{k}) = \mathbf{k}^2 + m^2$. Furthermore, from (A3.3) we easily verify that the hermitization matrix is

$$\eta = \gamma^0 \tag{10.3.7}$$

The free field for the Dirac equation is given by

$$\psi^0(x) = \sum_r \int d^3k \left[u^r(k) a^r(k) e^{i(k\cdot x - \varepsilon_k t)} + v^r(k) b^{r\dagger}(k) e^{-i(k\cdot x - \varepsilon_k t)} \right] \tag{10.3.8}$$

so that

$$\tilde{\psi}^0(x) = \sum_r \int d^3k \left[u^{r*}(k) \tilde{a}^r(k) e^{-i(k\cdot x - \varepsilon_k t)} + v^{r*}(k) \tilde{b}^{r\dagger}(k) e^{i(k\cdot x - \varepsilon_k t)} \right] \tag{10.3.9}$$

Here $\varepsilon_k \equiv \varepsilon(k)$. The operators* $a^r(k)$, $\tilde{a}^r(k)$ and $b^r(k)$, $\tilde{b}^r(k)$ are related to the physical annihilation and creation operators through the Bogoliubov transformation :

$$a^r(k) = c(\varepsilon_k) a^r(k,\beta) + d(\varepsilon_k) \tilde{a}^{\dagger r}(k,\beta) \tag{10.3.10a}$$

$$\tilde{a}^r(k) = c(\varepsilon_k) \tilde{a}^r(k,\beta) - d(\varepsilon_k) a^{\dagger r}(k,\beta) \tag{10.3.10b}$$

$$b^r(k) = c(\varepsilon_k) b^r(k,\beta) - d(\varepsilon_k) \tilde{b}^{r\dagger}(k,\beta) \tag{10.3.11a}$$

$$\tilde{b}^r(k) = c(\varepsilon_k) \tilde{b}^r(k,\beta) + d(\varepsilon_k) b^{r\dagger}(k,\beta) \tag{10.3.11b}$$

where $c(\varepsilon_k)$ and $d(\varepsilon_k)$ are given by (6.2.4). The amplitudes $u^r(k)$ and $v^r(k)$ satisfy the following sum rules :

$$\frac{(2\pi)^3}{h} \sum_r u_r(k) \bar{u}_r(k) = \frac{\gamma^0 \varepsilon - \gamma\cdot k + m}{2\varepsilon} \tag{10.3.12}$$

$$\frac{(2\pi)^3}{h} \sum_r v_r(k) \bar{v}_r(k) = \frac{\gamma^0 \varepsilon - \gamma\cdot k - m}{2\varepsilon} \tag{10.3.13}$$

Equations (10.3.12, 13) are obtained from (A7.3 and A7.4). The two-point function is calculated from

---

* The operator $a^r(k)$ annihilates electrons and $b^r(k)$ annihilates positrons.

$$S_c^{\alpha\beta}(x-y) = \langle 0(\beta)|T\,\psi^{\alpha}(x)\overline{\psi}^{\beta}(y)|0(\beta)\rangle$$
$$= \theta(t_x - t_y)\langle 0(\beta)|\psi^{\alpha}(x)\overline{\psi}^{\beta}(y)|0(\beta)\rangle \qquad (10.3.14)$$
$$-\theta(t_y - t_x)\langle 0(\beta)|\overline{\psi}^{\beta}(y)\psi^{\alpha}(x)|0(\beta)\rangle$$

where $\overline{\psi}(y) = \psi^{\dagger}(y)\gamma^0$. The Fourier amplitude is given by

$$S_c^{\alpha\beta}(x-y) = \frac{\hbar}{(2\pi)^4}\int d^4k\, e^{-ik\cdot(x-y)}S_c^{\alpha\beta}(k) \qquad (10.3.15)$$

Thus for example, when we let $z = x - y$ then

$$S_{c\mu\nu}^{11}(k) = \frac{1}{\hbar}\int d^4z\, e^{ik\cdot(x-y)}S_{c\mu\nu}^{11}(x-y)$$
$$= \frac{1}{\hbar}\int d^4z\, e^{ik\cdot(x-y)}\{\theta(t_z)\langle 0(\beta)|\psi_{\mu}(x)\overline{\psi}_{\nu}(y)|0(\beta)\rangle - \theta(-t_z)\langle 0(\beta)|\overline{\psi}_{\nu}(y)\psi_{\mu}(x)|0(\beta)\rangle\} \qquad (10.3.16)$$

where μ and ν are spinor indices. Furthermore, when we use (10.3.8) we find

$$\langle 0(\beta)|\psi_{\mu}(x)\overline{\psi}_{\nu}(y)|0(\beta)\rangle$$
$$= \langle 0(\beta)|\sum_r \int d^3p\left[\, u_{\mu}^r(\mathbf{p})a^r(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - \varepsilon_p t_x)} + v_{\mu}^r(\mathbf{p})b^{r\dagger}(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{x} - \varepsilon_p t_x)}\right]$$
$$\times \sum_s \int d^3p'\left[\, u_{\nu}^s(\mathbf{p}')a^s(\mathbf{p}')e^{-i(\mathbf{p}'\cdot\mathbf{y} - \varepsilon_{p'}t_y)} + v_{\nu}^s(\mathbf{p}')b^{s\dagger}(\mathbf{p}')e^{i(\mathbf{p}'\cdot\mathbf{y} - \varepsilon_{p'}t_y)}\right]|0(\beta)\rangle \qquad (10.3.17)$$

The vacuum expectation values are evaluated using the Bogoliubov transformation (10.3.10a). For example

$$\langle 0(\beta)|\, a^r(\mathbf{p})a^{s\dagger}(\mathbf{p}')\,|0(\beta)\rangle = \langle 0(\beta)|\,(c(\varepsilon_p)a^r(\mathbf{p},\beta) + d(\varepsilon_p)\bar{a}^{\dagger}(\mathbf{p},\beta))$$
$$\times\,(c(\varepsilon_{p'})a^s(\mathbf{p}',\beta) + d(\varepsilon_{p'})\bar{a}^{s\dagger}(\mathbf{p}',\beta))|0(\beta)\rangle \qquad (10.3.18)$$

Since the tilde and non-tilde operators commute, and $a^r(\mathbf{p},\beta)$ annihilates the vacuum then we get

$$\begin{aligned}
&\langle 0(\beta)|\, c(\varepsilon_p)c(\varepsilon_{p'})a^r(\mathbf{p},\beta)a^{s\dagger}(\mathbf{p}',\beta)\,|0(\beta)\rangle \\
&= \langle 0(\beta)|\, c(\varepsilon_p)c(\varepsilon_{p'})[\delta(\mathbf{p}-\mathbf{p}')\delta^{rs} - a^{s\dagger}(\mathbf{p}',\beta)a^r(\mathbf{p},\beta)\,|0(\beta)\rangle \\
&= \delta(\mathbf{p}-\mathbf{p}')\delta^{rs}c(\varepsilon_p)c(\varepsilon_{p'})
\end{aligned} \tag{10.3.19}$$

where we used the commutation relation between $a^r(\mathbf{p})$ and $a^{s\dagger}(\mathbf{p}')$. Evaluating the other expectation values in the same way we get immediately

$$\begin{aligned}
\langle 0(\beta)|\psi_\mu(x)\overline{\psi}_\nu(y)|0(\beta)\rangle = \sum_r \int d^3k \Big[\, & c^2(\varepsilon_p)u^r_\mu(\mathbf{p})\overline{u}^r_\nu(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{z}-\varepsilon_p t_z)} \\
& + d^2(\varepsilon_p)v^r_\mu(\mathbf{p})\overline{v}^r_\nu(\mathbf{p})e^{-i(\mathbf{p}\cdot\mathbf{z}-\varepsilon_p t_z)}\Big]
\end{aligned} \tag{10.3.20}$$

When we use the sum rules (10.3.12,13) we find for the Fourier amplitude of (10.3.20) :

$$\begin{aligned}
\frac{1}{(2\pi)^3}\int d^4z\, e^{ik_0 t_z}e^{-i\mathbf{k}\cdot\mathbf{z}}\theta(t_z)\int d^3k \Bigg[& c^2(\varepsilon_p)\left(\frac{\gamma^0\varepsilon-\boldsymbol{\gamma}\cdot\mathbf{p}+m}{2\varepsilon}\right)_{\mu\nu} e^{i(\mathbf{p}\cdot\mathbf{z}-\varepsilon_p t_z)} \\
& + d^2(\varepsilon_p)\left(\frac{\gamma^0\varepsilon-\boldsymbol{\gamma}\cdot\mathbf{p}-m}{2\varepsilon}\right)_{\mu\nu} e^{-i(\mathbf{p}\cdot\mathbf{z}-\varepsilon_p t_z)}\Bigg]
\end{aligned} \tag{10.3.21}$$

The d^3z integration gives

$$\begin{aligned}
\int dt_z\, e^{ik_0 t_z}\,\theta(t_z)\int d^3k \Bigg[& \delta(\mathbf{p}-\mathbf{k})c^2(\varepsilon_p)\left(\frac{\gamma^0\varepsilon-\boldsymbol{\gamma}\cdot\mathbf{p}+m}{2\varepsilon}\right)_{\mu\nu} e^{-i\varepsilon_p t_z} \\
& + \delta(\mathbf{p}+\mathbf{k})d^2(\varepsilon_p)\left(\frac{\gamma^0\varepsilon-\boldsymbol{\gamma}\cdot\mathbf{p}-m}{2\varepsilon}\right)_{\mu\nu} e^{i\varepsilon_p t_z}\Bigg]
\end{aligned} \tag{10.3.22}$$

Integrating over d^3k gives

$$\int dt_z\, e^{ik_0 t_z}\,\theta(t_z)\left[\left(\frac{\gamma^0\varepsilon-\gamma\cdot k+m}{2\varepsilon}\right)_{\mu\nu} c^2(\varepsilon_k)e^{-i\varepsilon_k t_z}+\left(\frac{\gamma^0\varepsilon+\gamma\cdot k-m}{2\varepsilon}\right)_{\mu\nu} d^2(\varepsilon_k)e^{i\varepsilon_k t_z}\right]$$

$$=\int_0^{\infty} dt_z\left[\left(\frac{\gamma^0\varepsilon-\gamma\cdot k+m}{2\varepsilon}\right)_{\mu\nu} c^2(\varepsilon_k)e^{i(k_0-\varepsilon_k)t_z}+\left(\frac{\gamma^0\varepsilon+\gamma\cdot k-m}{2\varepsilon}\right)_{\mu\nu} d^2(\varepsilon_k)e^{i(k_0+\varepsilon_k)t_z}\right]$$

$$=-i\left[\frac{\gamma^0\varepsilon-\gamma\cdot k+m}{2\varepsilon}\frac{c^2(\varepsilon_k)}{k_0-\varepsilon+i\delta}+\frac{\gamma^0\varepsilon+\gamma\cdot k-m}{2\varepsilon}\frac{d^2(\varepsilon_k)}{k_0-\varepsilon+i\delta}\right] \tag{10.3.23}$$

Evaluating the second term in (10.3.17) leads to

$$\begin{aligned} i\,S_c^{11}(k) = &\frac{\gamma^0\varepsilon-\gamma\cdot k+m}{2\varepsilon}\left[\frac{c^2(\varepsilon_k)}{k_0-\varepsilon+i\delta}+\frac{d^2(\varepsilon_k)}{k_0-\varepsilon+i\delta}\right] \\ &+\frac{\gamma^0\varepsilon+\gamma\cdot k-m}{2\varepsilon}\left[\frac{d^2(\varepsilon_k)}{k_0+\varepsilon+i\delta}+\frac{c^2(\varepsilon_k)}{k_0-\varepsilon-i\delta}\right] \end{aligned} \tag{10.3.24}$$

The other components of (10.3.15) are evaluated in the same way and they can all be put in the following simple form :

$$\begin{aligned} S_{c\mu\nu}^{\alpha\beta}(k) = &\left(\frac{\gamma^0\varepsilon-\gamma\cdot k+m}{2\varepsilon}\right)_{\mu\nu}[U(\varepsilon)(k_0-\varepsilon+i\delta\tau)^{-1}U^{\dagger}(\varepsilon)]_{\alpha\beta} \\ &+\left(\frac{\gamma^0\varepsilon+\gamma\cdot k-m}{2\varepsilon}\right)_{\mu\nu}[U(-\varepsilon)(k_0+\varepsilon+i\delta\tau)^{-1}U^{\dagger}(-\varepsilon)]_{\alpha\beta} \end{aligned} \tag{10.3.25}$$

where $\varepsilon = \varepsilon_k$.

In writing down the second term of (10.3.25) we used $c(\varepsilon) = d(-\varepsilon)$ and $d(\varepsilon) = c(-\varepsilon)$. For the calculation of the Lamb shift, it will be interesting if we can separate the zero temperature result from the finite temperature result. In other words, we wish to write the Lamb shift as the zero temperature result plus the finite temperature corrections. This is achieved by separating the propagators (10.2.6, 10.3.25) into temperature independent and temperature dependent parts as :

$$\Delta(k) = \Delta_{(0)}(k) + \Delta_{(\beta)}(k)$$
$$S(k) = S_{(0)}(k) + S_{(\beta)}(k) \qquad (10.3.26)$$

where $S_{(0)}$, $\Delta_{(0)}$ are temperature independent and $S_{(\beta)}$, $\Delta_{(\beta)}$ are temperature dependent.

This separation is shown in appendix C. The results are :

$$\Delta_{(0)}(k) = \tau[k_0^2 - (\omega - i\delta\tau)^2]^{-1} \qquad (10.3.27a)$$

$$\Delta_{(\beta)}(k) = \frac{-2\pi i\delta(k_0^2 - \omega^2)}{e^{\beta|k_0|} - 1}\begin{pmatrix} 1 & e^{\beta k_0/2} \\ e^{\beta k_0/2} & 1 \end{pmatrix} \qquad (10.3.27b)$$

for the photon propagator and

$$S_{(0)}(k) = \frac{\gamma^0\varepsilon - \gamma\cdot\mathbf{k} + m}{2\varepsilon}(k_0 - \varepsilon + i\delta\tau)^{-1} + \frac{\gamma^0\varepsilon + \gamma\cdot\mathbf{k} - m}{2\varepsilon}(k_0 + \varepsilon - i\delta\tau)^{-1} \qquad (10.3.28a)$$
$$= \frac{\not{k} + m}{k_0^2 - (\varepsilon - i\delta\tau)^2} = \frac{\not{k} + m}{k^2 - m^2 + i\delta\tau}.$$

$$S_{(\beta)}(k) = \frac{2\pi i}{e^{\beta|k_0|} + 1}\left[\frac{\gamma^0\varepsilon - \gamma\cdot\mathbf{k} + m}{2\varepsilon}\begin{pmatrix} 1 & e^{\beta|k_0|/2} \\ e^{\beta|k_0|/2} & -1 \end{pmatrix}\delta(k_0 - \varepsilon)\right. \qquad (10.3.28b)$$
$$\left. + \frac{\gamma^0\varepsilon + \gamma\cdot\mathbf{k} - m}{2\varepsilon}\begin{pmatrix} -1 & e^{\beta|k_0|/2} \\ e^{\beta|k_0|/2} & 1 \end{pmatrix}\delta(k_0 + \varepsilon)\right]$$

for the Dirac propagator.

The Electron Self-Energy, Vacuum Polarization and the Vertex

When evaluating the Lamb shift, it is found that the shift can be separated into two parts : one which corresponds to the continuum as obtained from the radiative corrections to Coulomb scattering and the second which corresponds to discrete energy as obtained from the bound states of the atom. The radiative corrections require that we renormalize the electron and photon self energy and also a third order diagram called the vertex, as the contribution to the Lamb shift from the continuum comes from these diagrams. Before we proceed with this, let us discuss how the Feynman diagrams are modified in the thermal situation.

Feynman Rules

The interaction Lagrangian for electrons and photons in zero temperature QFT is

$$\mathfrak{I}_I(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \qquad (11.1.1)$$

In TFD, this is replaced by

$$\widehat{\mathfrak{I}}_I(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) + e\tilde{\bar{\psi}}(x)\gamma^\mu\tilde{\psi}(x)\tilde{A}_\mu(x) \qquad (11.1.2)$$

However it is simpler to write this in the thermal doublet notation :

$$\widehat{\mathfrak{I}}_I(x) = -e\sum_\alpha \bar{\psi}^\alpha(x)\gamma^\mu\psi^\alpha(x)A_\mu^\alpha(x) \qquad (11.1.3)$$

With this, we can easily construct Feynman rules based on this Lagrangian in the same way as zero temperature field theory. In other words, every field in the zero temperature field

theory is now replaced by the thermal doublet so that the Gell-Mann-Low formula** becomes

$$\langle 0(\beta)|T\varsigma_1^{\alpha_1}(x_1)\cdots\varsigma_n^{\alpha_n}(x_n)|0(\beta)\rangle = \frac{{}_\beta\langle T\varsigma_1^{\alpha_1}(x_1)\cdots\varsigma_n^{\alpha_n}(x_n)\exp\{i\int d^4x\ \widehat{\mathfrak{X}}_I(x)\}\rangle_\beta}{{}_\beta\langle T\exp\{i\int d^4x\ \widehat{\mathfrak{X}}_I(x)\}\rangle_\beta} \tag{11.1.4}$$

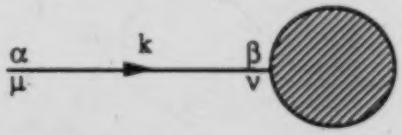
where the left hand side is in the Heisenberg picture and the expectation value on the right hand side is in the interaction picture. The vertices are still the same as in the zero temperature field theory, however the propagators (at least the two point functions) can be expressed as 2 x 2 matrices. In general, one must resort to (11.1.4) to obtain the correct expression for a diagram. The Feynman rules are established for any field theory in this way. Table 11-1 summarizes the Feynman rules that we use here. The photon is denoted by wavy lines and the electron by a solid line. Notice that as in zero temperature field theory, there is a spinor index associated with every corner. However in the case of TFD, there is also a thermal index associated with every corner. This has the effect of replacing the propagators (10.1.1,2) by the thermal propagators (10.1.6, 10.25). Furthermore, we notice that the vertices are the same as in the zero temperature field theory.

** H. Matsumoto, I. Ojima and H. Umezawa, *Ann. Phys.* **152**, (1984) 348.

Table 11-1 : Feynman rules for spinor electrodynamics in TFD*

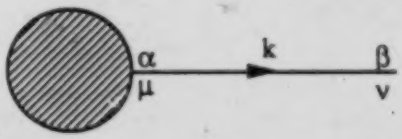
1. *External Lines*

Incoming fermion



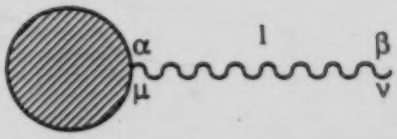
$S^{\alpha\beta}_{\mu\nu}(k)$

Outgoing fermion



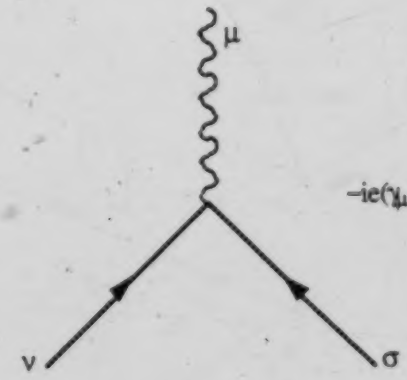
$S^{\alpha\beta}_{\mu\nu}(-k)$

Incident photon



$\Delta^{\alpha\beta}_{\mu\nu}(l)$

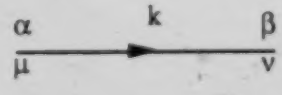
2. *Vertices*



$-ie(\gamma_\mu)_{\lambda\sigma}(2\pi)^4\delta^4(\Sigma \text{ incoming momenta})$

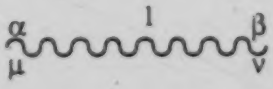
3. *Propagators (internal lines)*

Fermion



$\frac{d^4k}{(2\pi)^4} S^{\alpha\beta}_{\mu\nu}(k)$

Photon



$\frac{d^4l}{(2\pi)^4} \Delta^{\alpha\beta}_{\mu\nu}(l)$

* We will use the convention of letting μ,ν,σ stand for spinor indices and α,β,γ for thermal indices

The Electron Self-Energy

The electron self energy with two external lines is illustrated in fig. (11.1).

Fig. 11.1 The electron self energy to second (lowest) order

This diagram can be inserted into an external or an internal line of any diagram (fig. 11.2). When this is inserted into the latter, it has the effect of replacing the electron propagation function by $S^{\alpha\beta}(k)$ by another function. For example, to second order :

$$S'^{\alpha\delta}(k) = S^{\alpha\delta}(k) + S^{\alpha\beta}(k)\Sigma^{\beta\gamma}(k)S^{\gamma\delta}(k) \qquad (11.2.1)$$

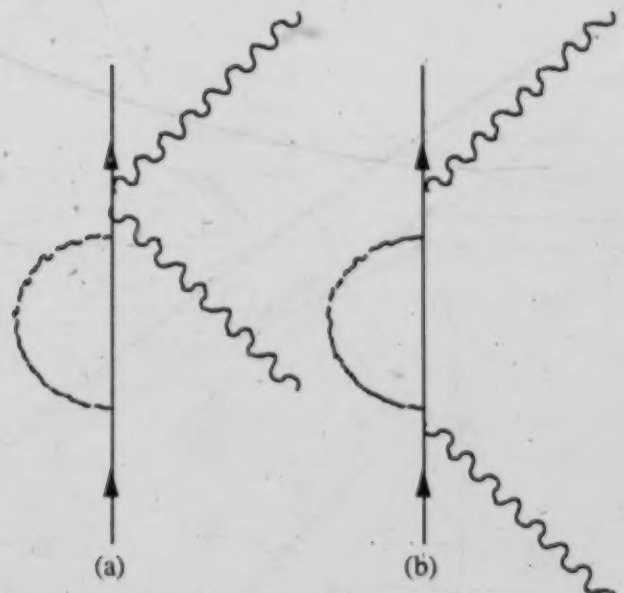


Fig. 11.2 Examples of second order self-energy parts inserted into (a) external and (b) internal electron lines of a Compton diagram.

The self energy $\Sigma^{\beta\gamma}(k)$ (to lowest order) is obtained by using table 11-1 :

$$\Sigma^{\alpha\beta}(k) = \frac{-ie^2}{(2\pi)^4}\int d^4l\, \gamma_\nu S^{\alpha\beta}(k-l)\gamma^\nu \Delta^{\alpha\beta}(l) \qquad (11.2.2)$$

Since S_β and Δ_β have an exponential damping factor, terms containing them in (11.2.2) do not diverge for high momenta. Thus the divergence arises in the zero temperature part of (11.2.2). we rewrite (11.2.2) as :

$$\Sigma^{\alpha\delta}(p) = \Sigma^{\alpha\delta}_{(0)}(p) + \Sigma^{\alpha\delta}_{(\beta)}(p) \qquad (11.2.3)$$

where

$$\Sigma^{\alpha\beta}_{(0)}(p) = \frac{-ie^2}{(2\pi)^4}\int d^4l\, \gamma_\nu S_0^{\alpha\beta}(k-l)\gamma^\nu \Delta_0^{\alpha\beta}(l) \qquad (11.2.4)$$

and

$$\Sigma^{\alpha\beta}_{(\beta)}(p) = \frac{-ie^2}{(2\pi)^4}\int d^4l\, [\, \gamma_\nu S^{\alpha\beta}_{(0)}(k-l)\gamma^\nu \Delta^{\alpha\beta}_{(\beta)}(l) + \gamma_\nu S^{\alpha\beta}_{(\beta)}(k-l)\gamma^\nu \Delta^{\alpha\beta}_{(0)}(l)\gamma_\nu S^{\alpha\beta}_{(\beta)}(k-l)\gamma^\nu \Delta^{\alpha\beta}_{(\beta)}(l)\,] \qquad (11.2.5)$$

The renormalization is performed from the same prescription as in zero temperature field theory. First we extract the mass counterterm (fig. 11.3) from

$$\Sigma^{\alpha\beta}(\not{k} = m) + \delta m = 0 \qquad (11.2.6)$$

Fig. 11.3 The mass counterterm is indicated by a cross in the electron line.

The zero temperature parts of the self energy are the same as the usual results and can be found in any standard text on QFT. So we will quote the pertinent results and explain any extensions to TFD. The zero temperature part Σ_0 is expanded as :

$$\Sigma_{(0)}(k) = \delta m_{(0)} + (\not{k} - m)Z_{(0)} + (\not{k} - m)^2\Sigma_{f(0)}(k) \qquad (11.2.7)$$

The divergent part is contained in the constants δm_0 and Z_0. These are given by :

$$\delta m_{(0)} = \frac{\alpha}{2\pi} m \left(\frac{3}{2} D + \frac{9}{4}\right) \tag{11.2.8}$$

$$Z_{(0)} = \frac{\alpha}{4\pi} \left(D - 4 \int \frac{dx}{x} + \frac{11}{2}\right) \tag{11.2.9}$$

$$\Sigma_{\ell(0)}(k) = \frac{\alpha}{2\pi m} \left\{ \frac{1}{2(1-\rho)} \left(1 - \frac{2-3\rho}{1-\rho} \ln \rho\right) + \frac{k+m}{m} \left[\frac{1}{2\rho(1-\rho)} \left(2 - \rho + \frac{-4+4\rho+\rho^2}{1-\rho} \ln \rho\right) - \frac{2}{\rho} \int_0^1 dx \left(\frac{1}{x} - x\right) \right] \right\} \tag{11.2.10}$$

The constant D is a divergent integral given by :

$$D = \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2 + m^2)^2} \tag{11.2.11}$$

In the above expressions, we have omitted the thermal index because they are simply multiplied by the unit matrix. We will use this convention of omitting the thermal indices whenever the result is diagonal and multiplied by the unit matrix. The zero temperature part of Σ is real. On the other hand, $\Sigma_\beta(k)$ will have real and imaginary parts, which we show separately because the real parts are diagonal, and the imaginary parts can be put into a compact form on their own. This separation is done by using

$$\frac{1}{x - x_0 \pm i\delta} = \frac{P}{x - x_0} \mp i\pi\delta(x - x_0) \tag{11.2.12}$$

The result of this somewhat lengthy calculation is :

$$\mathrm{Re}\Sigma_{(\beta)}(k) = \frac{\alpha}{\pi^2}\int \frac{d^3l}{e^{\beta\omega}-1}\left\{\frac{\not{k}+\not{\gamma}\cdot l-2m}{2\omega}\left[\frac{1}{k_0^2-(\varepsilon+\omega)^2}+\frac{1}{k_0^2-(\varepsilon-\omega)^2}\right]\right.$$
$$\left.-\frac{\not{\gamma}\cdot(k-l)-2m}{2\varepsilon}\left[\frac{1}{k_0^2-(\varepsilon+\omega)^2}+\frac{1}{k_0^2-(\varepsilon-\omega)^2}\right]\right\}$$
$$-\frac{\alpha}{\pi^2}\int \frac{d^3l}{e^{\beta\varepsilon}+1}\left\{\frac{\not{k}+\not{\gamma}\cdot l-2m}{2\omega}\left[\frac{1}{k_0^2-(\varepsilon+\omega)^2}-\frac{1}{k_0^2-(\varepsilon-\omega)^2}\right]\right.$$
$$\left.-\frac{\not{\gamma}\cdot(k-l)-2m}{2\varepsilon}\left[\frac{1}{k_0^2-(\varepsilon+\omega)^2}+\frac{1}{k_0^2-(\varepsilon-\omega)^2}\right]\right\} \tag{11.2.13}$$

$$\mathrm{Im}\,\Sigma^{\alpha\beta}_{(\beta)}(k) = \frac{\alpha}{\pi}\int_{-\infty}^{\infty} d\kappa \int \frac{d^3l}{2\varepsilon}\{[\gamma^0\varepsilon-\not{\gamma}\cdot(k-l)-2m]\,\sigma'(\kappa;\varepsilon,\omega)$$
$$-[\gamma^0\varepsilon+\not{\gamma}\cdot(k-l)+2m]\,\sigma'(\kappa;-\varepsilon,\omega)]\,[U(\kappa)\tau U^{\dagger}(\kappa)]_{\alpha\beta} \tag{11.2.14}$$

where $\varepsilon = \varepsilon(k-l) = (k-l)^2 + m^2$

Now we immediately find

$$\mathrm{Re}\,\delta m_{(\beta)}(k) = \frac{\alpha}{\pi^2}\int d^3l\left\{\left[\frac{\not{\gamma}\cdot l-m}{2\omega(e^{\beta\omega}-1)}+\frac{\gamma^0k_0-\not{\gamma}\cdot l+m}{2\varepsilon(e^{\beta\varepsilon}+1)}\right]\left[\frac{1}{2k\cdot l-2\varepsilon\omega}+\frac{1}{2k\cdot l+2\varepsilon\omega}\right]\right.$$
$$\left.-\left[\frac{\not{\gamma}\cdot l-m}{2\omega(e^{\beta\varepsilon}+1)}+\frac{\gamma^0k_0-\not{\gamma}\cdot l+m}{2\varepsilon(e^{\beta\omega}-1)}\right]\left[\frac{1}{2k\cdot l-2\varepsilon\omega}-\frac{1}{2k\cdot l+2\varepsilon\omega}\right]\right\} \tag{11.2.15}$$

$$\mathrm{Im}\,\delta m^{\alpha\beta}_{(\beta)}(k) = \frac{\alpha}{\pi}\int_{-\infty}^{\infty} d\kappa \int \frac{d^3l}{2\varepsilon}[\{\gamma^0\varepsilon-\gamma^0k_0+\not{\gamma}\cdot l-m\}\sigma'(\kappa;\varepsilon,\omega)$$
$$-\{\gamma^0\varepsilon+\gamma^0k_0-\not{\gamma}\cdot l+m\}\sigma'(\kappa;-\varepsilon,\omega)]\,[U(\kappa)\tau U^{\dagger}(\kappa)]_{\alpha\beta} \tag{11.2.16}$$

The form of the imaginary part of the self-energy is suggested from the product rule of a fermion and boson two point function (B1.7). To determine the part of the self energy that vanishes when $\not{k} = m$ we subtract the mass counterterm from the self energy so that by writing

$$\Sigma_{(\beta)}(k) = \delta m_{(\beta)}(k) + (\not{k} - m)Z_{(\beta)}(k) + (\not{k} - m)^2\Sigma_{\ell(\beta)}(k) + \Sigma'_{\ell(\beta)}(k) \qquad (11.2.17)$$

we find

$$\begin{aligned}
\mathrm{Re}\, Z_{(\beta)}(k) = \frac{\alpha}{\pi^2}\int d^3l \Bigg\{ & \left[\frac{1}{k_0^2 - (\varepsilon+\omega)^2}\left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{\beta\varepsilon}+1}\right)\frac{1}{2\omega}\right. \\
& \left. + \frac{1}{k_0^2 - (\varepsilon-\omega)^2}\left(\frac{1}{e^{\beta\omega}-1} + \frac{1}{e^{\beta\varepsilon}+1}\right)\right] \\
& - 2m(\gamma\cdot l - m)\int_0^1 dz \frac{1}{[(k^2-m^2)z + 2k\cdot l - 2\varepsilon\omega]^2}\left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{\beta\varepsilon}+1}\right)\left(\frac{1}{2\omega} + \frac{1}{2\varepsilon}\right) \\
& - 2m(\gamma\cdot l - m)\int_0^1 dz \frac{1}{[(k^2-m^2)z + 2k\cdot l + 2\varepsilon\omega]^2}\left(\frac{1}{e^{\beta\omega}-1} + \frac{1}{e^{\beta\varepsilon}+1}\right)\left(\frac{1}{2\omega} - \frac{1}{2\varepsilon}\right)\Bigg\}
\end{aligned} \qquad (11.2.18)$$

$$\begin{aligned}
\mathrm{Re}\, \Sigma_{\ell(\beta)}(k) = -\frac{\alpha}{\pi^2}\int d^3l \Bigg\{ & (\gamma\cdot l - m)\int_0^1 dz \frac{1}{[(k^2-m^2)z + 2k\cdot l - 2\varepsilon\omega]^2} \\
& \times \left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{\beta\varepsilon}+1}\right)\left(\frac{1}{2\omega} + \frac{1}{2\varepsilon}\right) \\
& - (\gamma\cdot l - m)\int_0^1 dz \frac{1}{[(k^2-m^2)z + 2k\cdot l + 2\varepsilon\omega]^2}\left(\frac{1}{e^{\beta\omega}-1} + \frac{1}{e^{\beta\varepsilon}+1}\right)\left(\frac{1}{2\omega} - \frac{1}{2\varepsilon}\right)\Bigg\}
\end{aligned} \qquad (11.2.19)$$

$$\mathrm{Re}\,\Sigma'_{f(\beta)}(k) = \frac{\alpha}{\pi^2}\int \frac{d^3l}{2\varepsilon}\left\{\left[\frac{\gamma^0 k_0}{2k\cdot l - 2\varepsilon\omega} - \frac{\gamma\cdot k + m}{k_0^2 - (\varepsilon+\omega)^2}\right]\left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{\beta\varepsilon}+1}\right)\right.$$
$$\left. - \left[\frac{\gamma^0 k_0}{2k\cdot l + 2\varepsilon\omega} - \frac{\gamma\cdot k + m}{k_0^2 - (\varepsilon-\omega)^2}\right]\left(\frac{1}{e^{\beta\omega}-1} + \frac{1}{e^{\beta\varepsilon}+1}\right)\right\} \tag{11.2.20}$$

$$\mathrm{Im}\, Z^{\alpha\beta}_{(\beta)}(k) = \frac{\alpha}{\pi}\int_{-}^{-} d\kappa \int \frac{d^3l}{2\varepsilon}\,[\sigma'(\kappa;\varepsilon,\omega) + \sigma'(\kappa;\varepsilon,\omega)][U(\kappa)\tau U^{\dagger}(\kappa)]_{\alpha\beta} \tag{11.2.21}$$

In establishing (11.2.18-20) we used

$$\frac{1}{\alpha^n} - \frac{1}{\beta^n} = -\int_0^1 dz\,\frac{n(\alpha-\beta)}{[(\alpha-\beta)z+\beta]^{n+1}} \tag{11.2.22}$$

and

$$k^2 - m^2 = 2m(\not{k} - m) + (\not{k} - m)^2 \tag{11.2.23}$$

Thus if we have the term

$$\gamma\cdot l(k^2 - m^2) = (k^2 - m^2)\gamma\cdot l$$

then

$$2m\gamma\cdot l(\not{k} - m) + \gamma\cdot l(\not{k} - m)^2 = 2m(\not{k} - m)\gamma\cdot l + (\not{k} - m)^2\gamma\cdot l$$

In other words, we can just as well write

$$\Sigma_{(\beta)}(k) = \delta m_{(\beta)}(k) + Z_{(\beta)}(k)(\not{k} - m) + \Sigma_{f(\beta)}(k)(\not{k} - m)^2 + \Sigma'_{f(\beta)}(k) \tag{11.2.24}$$

We will use either (11.2.24)) or (11.2.17) whichever one is convenient. The term Z(k) is called the wave function renormalization and Σ_f is the finite part of the self energy.

The Vacuum Polarization

The vacuum polarization diagram or equivalently the photon self-energy is illustrated in fig (11.4). This diagram can be inserted into either internal or external photon lines (fig. 11.5). When inserted into the former, the propagation function $\Delta_{\mu\nu}(k) = g_{\mu\nu}\Delta(k)$ is replaced by

$$\Delta'_{\mu\nu}(k) = g_{\mu\nu}\Delta(k) + \Delta(k)\Pi_{\mu\nu}(k)\Delta(k) \qquad (11.3.1)$$

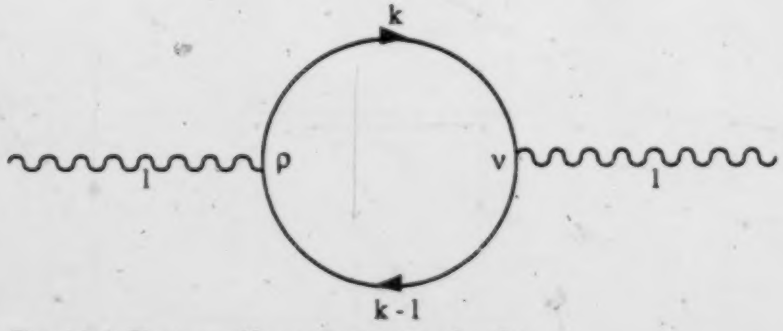


Fig. 11.4 Photon self-energy to second order.

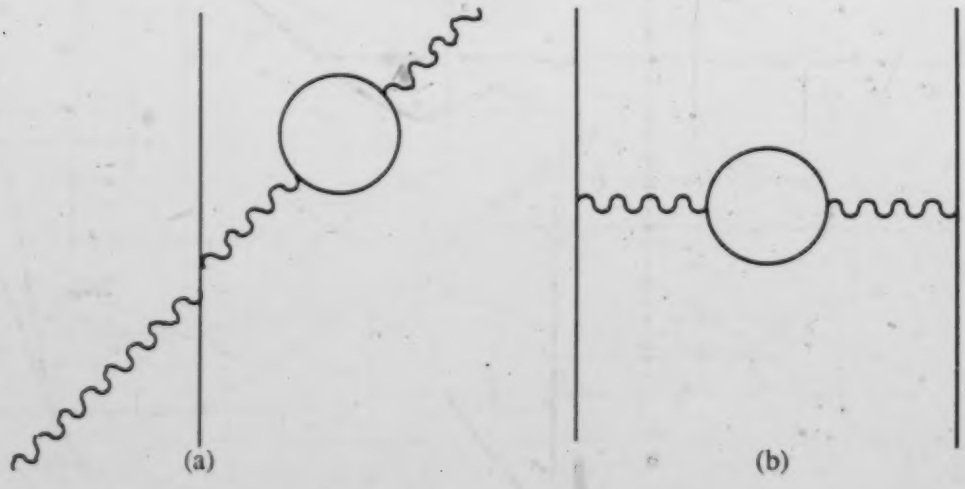


Fig. 11.5 Examples of photon-self energy parts inserted into (a) external and (b) internal photon lines.

From table 11-1 we find :

$$\Pi^{\alpha\beta}_{\mu\nu}(k) = \frac{-ie^2}{(2\pi)^4} \mathrm{Tr}\left\{\int d^4l\, \gamma_\mu S^{\alpha\beta}(l-k)\gamma_\nu S^{\beta\alpha}(l)\right\} \qquad (11.3.2)$$

The zero temperature part of (11.3.2) is denoted by $\Pi^{\alpha\beta}_{\mu\nu 0}$ and gives the usual zero temperature result multiplied by the unit matrix :

$$\Pi_{\mu\nu(0)}(k) = g_{\mu\nu}\Pi_{(0)}(k^2) \qquad (11.3.3)$$

where

$$\Pi_{(0)}(k^2) = k^2(-C + k^2\Pi_f(k_2)) \qquad (11.3.4)$$

$$-C = \frac{-\alpha}{3\pi}\left(D + \frac{5}{6}\right) \qquad (11.3.5)$$

$$k^2\Pi_f(k^2) = \frac{\alpha}{3\pi}\left[\frac{5}{3} - \frac{1}{\rho} - \left(1 - \frac{1}{2\rho}\right)\sqrt{1 + \frac{1}{\rho}}\,\ln\frac{\sqrt{1 + 1/\rho} + 1}{\sqrt{1 + 1/\rho} - 1}\right] \qquad (11.3.6)$$

with $\rho = k^2/m^2$.

The temperature dependent part is given by

$$\mathrm{Re}\,\Pi_{\mu\nu(\beta)}(k) = \tau\frac{\alpha}{8\pi^2}\int d^4l \left\{\frac{1}{\varepsilon_{l-k}}\frac{(l-k)^2 + m^2}{((l-k)^2 + m^2)^2 + \delta^2}\frac{\sigma_{\mu\nu}}{e^{\beta\varepsilon_{l-k}} + 1}\right.$$
$$\left. + \frac{1}{\varepsilon_l}\frac{l^2 + m^2}{(l^2 + m^2)^2 + \delta^2}\frac{\sigma'_{\mu\nu}}{e^{\beta\varepsilon_l} + 1}\right\} \qquad (11.3.7)$$

$$\mathrm{Im}\,\Pi^{11}_{\mu\nu(\beta)}(k) = \mathrm{Im}\,\Pi^{22}_{\mu\nu(\beta)}(k) = \frac{\alpha}{8\pi^2}\int d^4l \left\{ \frac{1}{\varepsilon_{l-k}} \frac{\delta}{((l-k)^2+m^2)^2+\delta^2} \frac{\sigma_{\mu\nu}}{e^{\beta\varepsilon_{l-k}}+1} + \frac{1}{\varepsilon_l} \frac{\delta}{(l^2+m^2)^2+\delta^2} \frac{\sigma'_{\mu\nu}}{e^{\beta\varepsilon_l}+1} + \frac{2\pi}{\varepsilon_{l-k}\varepsilon_l} \sigma''_{\mu\nu} \frac{1}{e^{\beta\varepsilon_{l-k}}+1} \frac{1}{e^{\beta\varepsilon_l}+1} \right\} \tag{11.3.8}$$

$$\mathrm{Im}\,\Pi^{11}_{\mu\nu(\beta)}(k) = \mathrm{Im}\,\Pi^{22}_{\mu\nu(\beta)}(k) = \frac{\alpha}{\pi}\int d^4l \left\{ \frac{1}{\varepsilon_{l-k}\varepsilon_l} \sigma''_{-\mu\nu} \frac{e^{\beta\varepsilon_{l-k}/2}}{(e^{\beta\varepsilon_{l-k}}+1)} \frac{e^{\beta\varepsilon_l/2}}{(e^{\beta\varepsilon_l}+1)} \right\} \tag{11.3.9}$$

where

$$\begin{aligned}
\sigma_{\mu\nu} &= \mathrm{Tr}\,\{\gamma_\mu(\not{l}-\not{k}+m)\gamma_\nu[(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)-(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)]\},\\
\sigma'_{\mu\nu} &= \mathrm{Tr}\,\{\gamma_\mu[(\gamma^0\varepsilon_{l-k}-\gamma\cdot(l-k)+m)\delta(l_0-k_0-\varepsilon_{l-k})\\
&\quad -(\gamma^0\varepsilon_{l-k}+\gamma\cdot(l-k)-m)\delta(l_0-k_0+\varepsilon_{l-k})]\gamma_\nu(l+m)\},\\
\sigma''_{-\mu\nu} &= \mathrm{Tr}\,\{\gamma_\mu[(\gamma^0\varepsilon_{l-k}-\gamma\cdot(l-k)+m)\delta(l_0-k_0-\varepsilon_{l-k})\\
&\quad -(\gamma^0\varepsilon_{l-k}+\gamma\cdot(l-k)-m)\delta(l_0-k_0+\varepsilon_{l-k})]\gamma_\nu\\
&\times[(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)-(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)]\},\\
\sigma''_{\mu\nu} &= \mathrm{Tr}\,\{\gamma_\mu[(\gamma^0\varepsilon_{l-k}-\gamma\cdot(l-k)+m)\delta(l_0-k_0-\varepsilon_{l-k})\\
&\quad +(\gamma^0\varepsilon_{l-k}+\gamma\cdot(l-k)-m)\delta(l_0-k_0+\varepsilon_{l-k})]\gamma_\nu\\
&\times[(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)+(\gamma^0\varepsilon_l-\gamma\cdot l+m)\delta(l_0-\varepsilon)]\}.
\end{aligned} \tag{11.3.10}$$

where $\varepsilon_l = \varepsilon(l)$, $\varepsilon_{l-k} = \varepsilon(l-k)$

Notice that we have left the l_0 integrations so as to keep the expressions simple. For the same reason we leave the traces. They can be quickly evaluated though using the standard trace theorems.

The Vertex

Contrary to the electron and photon self-energy function, which are two-point functions, the vertex diagram is a three point function as illustrated in fig. (11.6). This diagram can be inserted into every corner of any diagram as shown in fig. (11.7). Doing so has the effect of replacing the γ_μ of the vertex by

$$\Gamma_\mu = \gamma_\mu + \Lambda_\mu(k,k') \qquad (11.4.1)$$

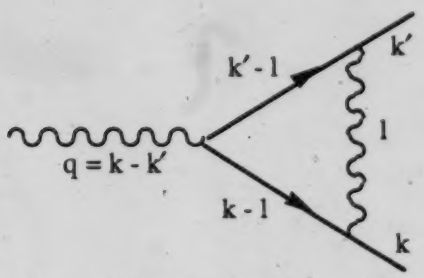


Fig. 11.6 Vertex diagram to second order.

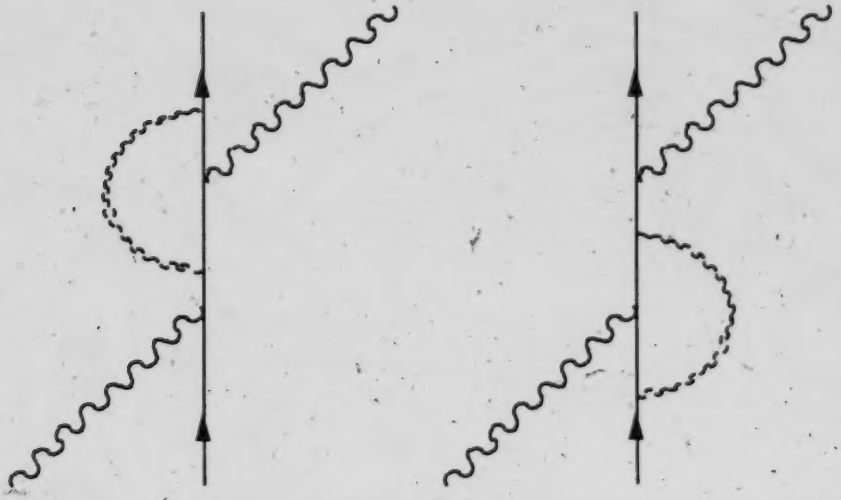


Fig. 11.7 Examples of second order vertex parts inserted into a Compton diagram.

where

$$\Lambda_\mu^{\alpha\beta\gamma}(k,k') = \frac{-ie^2}{(2\pi)^4}\int d^4l\, \gamma_\nu S^{\alpha\beta}(k'-l)\gamma_\mu S^{\gamma\alpha}(k-l)\gamma^\nu \Delta^{\gamma\beta}(l) \qquad (11.4.2)$$

For the zero temperature part of (11.4.2), there are only two components, Λ^{111}, Λ^{222} because the zero temperature part of the propagators are diagonal. The results are :

$$\Lambda_{\mu 0}(k,k') = L_0\gamma_\mu + \Lambda_{\mu f}(k,k') \qquad (11.4.3)$$

where

$$L_0 = \frac{\alpha}{4\pi}\left(D - 4\int_0^1 \frac{dx}{x} + \frac{11}{2}\right) \qquad (11.4.4)$$

$$\Lambda_{\mu f0}(k,k') = -\frac{\alpha}{2\pi}\int_0^1 dx \int_0^x dy \left\{\frac{K_\mu}{a^2} + \gamma_\mu \int_0^1 dz \frac{a^2 - m^2x^2}{m^2x^2 + (a^2 - m^2x^2)z} - 2m^2\gamma_\mu(1 - x - x^2/2)\frac{a^2 - m^2x^2}{a^2m^2x^2}\right\} \qquad (11.4.5)$$

and

$$\begin{aligned}
K_\mu(k,k',x,y) &= (1-x)(\not{k}' - m)\gamma_\mu(\not{k} - m) + \gamma_\mu C - (\not{k}' - m)M'_\mu \\
&\quad - M_\mu(\not{k} - m) - k_\mu m(1+x)(x-2y) + mx(1-x)i\sigma_{\mu\nu}k^\nu , \\
C &= k^2(1-x+y)(1-y) - (1-x)[(k^2+m^2)(1-x+y) + (k'^2+m^2)(1-y)] , \\
M_\mu &= \gamma_\mu m(1-x^2) - K_\mu(1-x)(1-x+y) + k_\mu(1+x-2y)(1-x+y) , \\
M'_\mu &= \gamma_\mu m(1-x^2) - K_\mu(1-x)(1-y) - k_\mu(1-x+2y)(1-y) , \\
\sigma_{\mu\nu} &= \frac{i}{2}[\gamma_\mu, \gamma_\nu] .
\end{aligned} \qquad (11.4.6)$$

The finite temperature part of the vertex function is denoted by $\Lambda_{\mu\beta}$. This can be separated into a similar way as the zero temperature part. Specifically, we can write

$$\Lambda_{\mu\beta}(k,k') = L_\mu(k,k') + \Lambda_{\mu\beta f}(k,k') \qquad (11.4.7)$$

Furthermore L_μ can be written as

$$L_\mu(k,k') = L(k,k')\gamma_\mu + L'_\mu(k,k') . \qquad (11.4.8)$$

However for simplicity we will not separate L_μ as in (11.4.8). The expressions for the various components for the vertex function are given over the next few pages. The sum over the spinor index ν is performed through :

$$\gamma_\nu(\not{k} - \not{l}' + m)\gamma_\mu(\not{k} - \not{l} + m)\gamma^\nu = -2(\not{k} - \not{l}')\gamma_\mu(\not{k} - \not{l}) + 4m(K_\mu - L_\mu) - 2m^2\gamma_\mu \qquad (11.4.9)$$

where $K_\mu = k_\mu + k'_\mu$ and $L_\mu = l_\mu + l'_\mu$. With this, the results of this lengthy calculation are :

$$\mathrm{Re}\,\Lambda^{111}_{\mu\beta}(k,k') = \mathrm{Re}\,\Lambda^{222}_{\mu\beta}(k,k') =$$

$$= -\frac{\alpha}{2\pi^2}\int d^4l\,\frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}\,N_\mu(k,k',l)\,\frac{(k'^2-m^2-2k'\cdot l)(k^2-m^2-2k\cdot l)-\delta^2}{[(k'^2-m^2-2k'\cdot l)^2+\delta^2][(k^2-m^2-2k\cdot l)^2+\delta^2]}$$

$$+\frac{\alpha}{8\pi^2}\int\frac{d^4l}{\varepsilon\omega}\,\frac{\sigma_\mu(\varepsilon,k',l)}{e^{\beta\varepsilon}+1}\,\frac{k'^2-m^2-2k'\cdot l-\delta^2}{(k'^2-m^2-2k'\cdot l)^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2}-\frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$+\frac{\alpha}{2\pi}\,\delta\int\frac{d^4l}{\varepsilon}\,\frac{\sigma_\mu(\varepsilon,k',l)}{e^{\beta\varepsilon}+1}\,\frac{1}{(k'^2-m^2-2k'\cdot l)^2+\delta^2}\,\frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$+\frac{\alpha}{8\pi^2}\int\frac{d^4l}{\varepsilon\omega}\,\frac{\sigma'_\mu(\varepsilon',k,l)}{e^{\beta\varepsilon'}+1}\,\frac{k^2-m^2-2k\cdot l-\delta^2}{(k^2-m^2-2k\cdot l)^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2}-\frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$+\frac{\alpha}{2\pi}\,\delta\int\frac{d^4l}{\varepsilon'}\,\frac{\sigma'_\mu(\varepsilon',k,l)}{e^{\beta\varepsilon'}+1}\,\frac{1}{(k^2-m^2-2k\cdot l)^2+\delta^2}\,\frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$+\frac{\alpha}{8\pi}\,\delta\int\frac{d^4l}{\varepsilon\varepsilon'\omega}\,\frac{\sigma_\mu(\varepsilon,\varepsilon',l)}{(e^{\beta\varepsilon'}+1)(e^{\beta\varepsilon}+1)}\left[\frac{1}{(l_0-\omega)^2+\delta^2}-\frac{1}{(l_0+\omega)^2+\delta^2}\right]$$

$$+\frac{\alpha}{2}\int\frac{d^4l}{\varepsilon\varepsilon'}\,\frac{\sigma_\mu(\varepsilon,\varepsilon',l)}{(e^{\beta\varepsilon'}+1)(e^{\beta\varepsilon}+1)}\,\frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

(11.4.10)

$$\mathrm{Re}\,\Lambda^{112}_{\mu\beta}(k,k') = \mathrm{Re}\,\Lambda^{221}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi}\int\frac{d^4l}{\varepsilon\varepsilon'}\,\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\,\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\,\sigma_{-\mu}(\varepsilon)\delta(l_0^2-\omega^2)$$

$$\times\frac{1}{(k'^2-m^2-2k'\cdot l)^2+\delta^2}$$

$$+\frac{\alpha}{2}\int\frac{d^4l}{\varepsilon\varepsilon'}\,\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\,\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\,\frac{1}{e^{\beta\varepsilon'}+1}\,\sigma_l(\varepsilon,\varepsilon')\delta(l_0^2-\omega^2)$$

(11.4.11)

$$\mathrm{Re}\,\Lambda^{121}_{\mu\beta}(k,k') = \mathrm{Re}\,\Lambda^{212}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi}\delta\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\sigma'_{-\mu}(\varepsilon')\delta(l_0^2-\omega^2)$$

$$\times\frac{1}{(k^2-m^2-2k\cdot l)^2+\delta^2}$$

$$+\frac{\alpha}{2}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\frac{1}{e^{\beta\varepsilon}+1}\sigma_2(\varepsilon,\varepsilon')\delta(l_0^2-\omega^2) \qquad (11.4.12)$$

$$\mathrm{Re}\,\Lambda^{122}_{\mu\beta}(k,k') = \mathrm{Re}\,\Lambda^{211}_{\mu\beta}(k,k') = \frac{\alpha}{8\pi}\delta\int \frac{d^4l}{\omega\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\sigma'_{-\mu}(\varepsilon',\varepsilon)$$

$$\times\left[\frac{1}{(l_0-\omega)^2+\delta^2}+\frac{1}{(l_0+\omega)^2+\delta^2}\right]$$

$$+\frac{\alpha}{2}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\frac{1}{e^{\beta\omega}-1}\sigma_{3\mu}(\varepsilon,\varepsilon')\delta(l_0^2-\omega^2) \qquad (11.4.13)$$

The imaginary parts are given by :

$$\mathrm{Im}\,\Lambda^{111}_{\mu\beta}(k,k') = -\mathrm{Im}\,\Lambda^{222}_{\mu\beta}(k,k') =$$

$$= \frac{\alpha}{2\pi^2}\delta \int d^4l \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1} N_\mu(k,k',l) \frac{k'^2 - 2m^2 - 2k'\cdot l + k^2 - 2k\cdot l}{[(k'^2 - m^2 - 2k'\cdot l)^2+\delta^2][(k^2 - m^2 - 2k\cdot l)^2+\delta^2]}$$

$$- \frac{\alpha}{8\pi^2}\delta \int \frac{d^4l}{\varepsilon\omega} \frac{\sigma_\mu(\varepsilon,k',l)}{e^{\beta\varepsilon}+1} \frac{1}{(k'^2 - m^2 - 2k'\cdot l)^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$- \frac{\alpha}{2\pi} \int \frac{d^4l}{\varepsilon} \frac{\sigma_\mu(\varepsilon,k',l)}{e^{\beta\varepsilon}+1} \frac{k'^2 - m^2 - 2k'\cdot l}{(k'^2 - m^2 - 2k'\cdot l)^2+\delta^2} \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$- \frac{\alpha}{8\pi^2}\delta \int \frac{d^4l}{\varepsilon'\omega} \frac{\sigma'_\mu(\varepsilon',k,l)}{e^{\beta\varepsilon'}+1} \frac{1}{(k^2 - m^2 - 2k\cdot l)^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$- \frac{\alpha}{2\pi} \int \frac{d^4l}{\varepsilon'} \frac{\sigma'_\mu(\varepsilon',k,l)}{e^{\beta\varepsilon'}+1} \frac{k^2 - m^2 - 2k\cdot l - \delta^2}{(k^2 - m^2 - 2k\cdot l)^2+\delta^2} \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$+ \frac{\alpha}{8\pi} \int \frac{d^4l}{\varepsilon\varepsilon'\omega} \frac{\sigma_\mu(\varepsilon,\varepsilon',l)}{(e^{\beta\varepsilon'}+1)(e^{\beta\varepsilon}+1)}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

(11.4.14)

$$\mathrm{Im}\,\Lambda^{112}_{\mu\beta}(k,k') = -\mathrm{Im}\,\Lambda^{221}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi} \int \frac{d^4l}{\varepsilon\varepsilon'} \frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1} \frac{e^{\beta\omega/2}}{e^{\beta\omega}-1} \sigma_{-\mu}(\varepsilon)\delta(l_0^2-\omega^2)$$

$$\times \frac{k'^2 - m^2 - 2k'\cdot l}{(k'^2 - m^2 - 2k'\cdot l)^2+\delta^2}$$

(11.4.15)

$$\mathrm{Im}\,\Lambda^{121}_{\mu\beta}(k,k') = -\mathrm{Im}\,\Lambda^{212}_{\mu\beta}(k,k') = -\frac{\alpha}{2\pi}\int \frac{d^4l}{\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\sigma'_{-\mu}(\varepsilon')\delta(l_0^2-\omega^2) \times \frac{k^2-m^2-2k\cdot l}{(k^2-m^2-2k\cdot l)^2+\delta^2} . \tag{11.4.16}$$

$$\mathrm{Im}\,\Lambda^{122}_{\mu\beta}(k,k') = -\mathrm{Im}\,\Lambda^{211}_{\mu\beta}(k,k') = \frac{\alpha}{8\pi}\int \frac{d^4l}{\omega\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\sigma'_{-\mu}(\varepsilon',\varepsilon) \times\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2}+\frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right] . \tag{11.4.17}$$

In the above expressions, we use the following function :

$$\begin{aligned}
&N_\mu(k,k',l) = -2(\not k - \not l)\gamma_\mu(\not k - \not l) + 4m(K_\mu - L_\mu) - 2m^2\gamma_\mu ,\\
&\sigma_\mu(\varepsilon,k',l) = N_\mu(\varepsilon,k',l)\delta(k_0 - l_0 - \varepsilon) + N_\mu(-\varepsilon,k',l)\delta(k_0 - l_0 + \varepsilon) ,\\
&N_\mu(\varepsilon,k',l) = -2(\not k' - \not l)\gamma_\mu(\gamma^0\varepsilon - \gamma\cdot k + \gamma\cdot l) + 4m(k'_\mu + \varepsilon g_\mu{}^0 - k_j g_\mu{}^j - l_\mu - l_j g_\mu{}^j) - 2m^2\gamma_\mu ,\\
&\sigma'_\mu(\varepsilon',k',l) = N'_\mu(\varepsilon',k',l)\delta(k'_0 - l_0 - \varepsilon') + N'_\mu(-\varepsilon',k',l)\delta(k'_0 - l_0 + \varepsilon') ,\\
&N'_\mu(\varepsilon',k',l) = -2(\gamma^0\varepsilon' - \gamma\cdot k' + \gamma\cdot l)\gamma_\mu(\not k - \not l) + 4m(k_\mu + \varepsilon' g_\mu{}^0 - k'_j g_\mu{}^j - l_\mu - l_j g_\mu{}^j) - 2m^2\gamma_\mu ,\\
&\sigma_\mu(\varepsilon,\varepsilon',l) = [N_\mu(\varepsilon,\varepsilon',l)\delta(k'_0 - l_0 - \varepsilon') + N_\mu(\varepsilon,-\varepsilon',l)\delta(k'_0 - l_0 + \varepsilon')]\delta(k_0 - l_0 - \varepsilon)\\
&\qquad + [N_\mu(-\varepsilon,\varepsilon',l)\delta(k'_0 - l_0 - \varepsilon') + N_\mu(-\varepsilon,-\varepsilon',l)\delta(k'_0 - l_0 + \varepsilon')]\delta(k_0 - l_0 + \varepsilon) ,\\
&N_\mu(\varepsilon,\varepsilon',l) = -2(\gamma^0\varepsilon' - \gamma\cdot k' + \gamma\cdot l)\gamma_\mu(\gamma^0\varepsilon - \gamma\cdot k + \gamma\cdot l) + 4m((\varepsilon+\varepsilon')g_\mu{}^0 - K_j g_\mu{}^j - 2l_j g_\mu{}^j)\\
&\qquad\qquad - 2m^2\gamma_\mu .
\end{aligned} \tag{11.4.18}$$

where $\varepsilon = \varepsilon(k-l)$, $\varepsilon' = \varepsilon(k'-l)$

When we separate the vertex function as (11.4.8), the function L_μ is uniquely defined by the relation

$$\Lambda_{\mu\beta}(k,k') = 0 \qquad \not k' = \not k = m \tag{11.4.18}$$

The separation is more evident when we rewrite (11.4.9) as

$$
\begin{aligned}
&-2(k' - l')\gamma_\mu(k - l) + 4m(K_\mu - L_\mu) - 2m^2\gamma_\mu \\
&= -2(k' - m)\gamma_\mu(k - m) + 2m\gamma_\mu(k' - m) + 2m(k - m)\gamma_\mu + 4m^2\gamma_\mu \\
&\quad + 2l'\gamma_\mu(k - m) + 2(k' - m)\gamma_\mu l - 2m\gamma_\mu l' - 2ml'\gamma_\mu - 2l'\gamma_\mu l
\end{aligned}
\tag{11.4.19}
$$

With this the functions in (11.4.18) are modified, so that we write for the function L_μ :

$$
\begin{aligned}
\mathrm{Re}\, L^{111}_{\mu\beta}(k,k') &= \mathrm{Re}\, L^{222}_{\mu\beta}(k,k') = -\frac{\alpha}{2\pi^2}\int d^4l\, \frac{\delta(l_0^2 - \omega^2)}{e^{\beta\omega} - 1} N_\mu \frac{4k\cdot k'l^2 - \delta^2}{[4k'^2l^2 + \delta^2][4k^2l^2 + \delta^2]} \\
&+ \frac{\alpha}{8\pi^2}\int \frac{d^4l}{\varepsilon\omega}\frac{\sigma_\mu(\varepsilon)}{e^{\beta\varepsilon} + 1}\frac{-2k'\cdot l - \delta^2}{4k'^2l^2 + \delta^2}\left[\frac{l_0 - \omega}{(l_0 - \omega)^2 + \delta^2} - \frac{l_0 + \omega}{(l_0 + \omega)^2 + \delta^2}\right] \\
&+ \frac{\alpha}{2\pi}\delta\int \frac{d^4l}{\varepsilon}\frac{\sigma_\mu(\varepsilon)}{e^{\beta\varepsilon} + 1}\frac{1}{4k'^2l^2 + \delta^2}\frac{\delta(l_0^2 - \omega^2)}{e^{\beta\omega} - 1} \\
&+ \frac{\alpha}{8\pi^2}\int \frac{d^4l}{\varepsilon\omega}\frac{\sigma'_\mu(\varepsilon')}{e^{\beta\varepsilon'} + 1}\frac{-2k\cdot l - \delta^2}{4k^2l^2 + \delta^2}\left[\frac{l_0 - \omega}{(l_0 - \omega)^2 + \delta^2} - \frac{l_0 + \omega}{(l_0 + \omega)^2 + \delta^2}\right] \\
&+ \frac{\alpha}{2\pi}\delta\int \frac{d^4l}{\varepsilon'}\frac{\sigma'_\mu(\varepsilon')}{e^{\beta\varepsilon'} + 1}\frac{1}{4k^2l^2 + \delta^2}\frac{\delta(l_0^2 - \omega^2)}{e^{\beta\omega} - 1} \\
&+ \frac{\alpha}{8\pi}\delta\int \frac{d^4l}{\varepsilon\varepsilon'\omega}\frac{\sigma_\mu(\varepsilon,\varepsilon')}{(e^{\beta\varepsilon'} + 1)(e^{\beta\varepsilon} + 1)}\left[\frac{1}{(l_0 - \omega)^2 + \delta^2} - \frac{1}{(l_0 + \omega)^2 + \delta^2}\right] \\
&+ \frac{\alpha}{2}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{\sigma_\mu(\varepsilon,\varepsilon')}{(e^{\beta\varepsilon'} + 1)(e^{\beta\varepsilon} + 1)}\frac{\delta(l_0^2 - \omega^2)}{e^{\beta\omega} - 1}
\end{aligned}
\tag{11.4.20}
$$

$$\mathrm{Re}\, L^{112}_{\mu\beta}(k,k') = \mathrm{Re}\, L^{221}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\sigma_{-\mu}(\varepsilon)\delta(l_0^2-\omega^2)$$

$$\times\frac{1}{4k'^2l^2+\delta^2}+\frac{\alpha}{2}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\frac{1}{e^{\beta\varepsilon'}+1}\sigma_1(\varepsilon,\varepsilon')\delta(l_0^2-\omega^2) \qquad (i1.4.21)$$

$$\mathrm{Re}\, L^{121}_{\mu\beta}(k,k') = \mathrm{Re}\, L^{212}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi}\delta\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\sigma'_{-\mu}(\varepsilon')\delta(l_0^2-\omega^2)$$

$$\times\frac{1}{k^2l^2+\delta^2}+\frac{\alpha}{2}\int \frac{d^4l}{\varepsilon\varepsilon'}\frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1}\frac{e^{\beta\omega/2}}{e^{\beta\omega}-1}\frac{1}{e^{\beta\varepsilon}+1}\sigma_2(\varepsilon,\varepsilon')\delta(l_0^2-\omega^2) \qquad (11.4.22)$$

$$\mathrm{Re}\, L^{122}_{\mu\beta}(k,k') = \mathrm{Re}\, L^{211}_{\mu\beta}(k,k') = \mathrm{Re}\, \Lambda^{122}_{\mu\beta}(k,k') \qquad (11.4.23)$$

The imaginary parts are given by :

$$\mathrm{Im}\, L^{111}_{\mu\beta}(k,k') = -\mathrm{Im}\, L^{222}_{\mu\beta}(k,k') = \frac{\alpha}{2\pi^2}\delta \int d^4l \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1} N_\mu \frac{-2k'\cdot l - 2k\cdot l}{[4k'^2l^2+\delta^2][4k^2l^2+\delta^2]}$$

$$-\frac{\alpha}{8\pi^2}\delta \int \frac{d^4l}{\varepsilon\omega} \frac{\sigma_\mu(\varepsilon)}{e^{\beta\varepsilon}+1} \frac{1}{4k'^2l^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$-\frac{\alpha}{2\pi} \int \frac{d^4l}{\varepsilon} \frac{\sigma_\mu(\varepsilon)}{e^{\beta\varepsilon}+1} \frac{-2k'\cdot l}{4k'^2l^2+\delta^2} \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$-\frac{\alpha}{8\pi^2}\delta \int \frac{d^4l}{\varepsilon'\omega} \frac{\sigma'_\mu(\varepsilon')}{e^{\beta\varepsilon'}+1} \frac{1}{4k^2l^2+\delta^2}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right]$$

$$-\frac{\alpha}{2\pi} \int \frac{d^4l}{\varepsilon'} \frac{\sigma'_\mu(\varepsilon')}{e^{\beta\varepsilon'}+1} \frac{-2k\cdot l}{4k^2l^2+\delta^2} \frac{\delta(l_0^2-\omega^2)}{e^{\beta\omega}-1}$$

$$+\frac{\alpha}{8\pi} \int \frac{d^4l}{\varepsilon\varepsilon'\omega} \frac{\sigma_\mu(\varepsilon)}{(e^{\beta\varepsilon'}+1)(e^{\beta\varepsilon}+1)}\left[\frac{l_0-\omega}{(l_0-\omega)^2+\delta^2} - \frac{l_0+\omega}{(l_0+\omega)^2+\delta^2}\right] \tag{11.4.24}$$

$$\mathrm{Im}\, L^{112}_{\mu\beta}(k,k') = -\mathrm{Im}\, L^{221}_{\mu\beta}(k,k') = -\frac{\alpha}{4\pi} \int \frac{d^4l}{\varepsilon\varepsilon'} \frac{e^{\beta\varepsilon/2}}{e^{\beta\varepsilon}+1} \frac{e^{\beta\omega/2}}{e^{\beta\omega}-1} \sigma_{-\mu}(\varepsilon)\delta(l_0^2-\omega^2) \times \frac{-2k'\cdot l}{k'^2l^2+\delta^2} \tag{11.4.25}$$

$$\mathrm{Im}\, L^{121}_{\mu\beta}(k,k') = -\mathrm{Im}\, L^{212}_{\mu\beta}(k,k') = -\frac{\alpha}{2\pi} \int \frac{d^4l}{\varepsilon'} \frac{e^{\beta\varepsilon'/2}}{e^{\beta\varepsilon'}+1} \frac{e^{\beta\omega/2}}{e^{\beta\omega}-1} \sigma'_{-\mu}(\varepsilon')\delta(l_0^2-\omega^2) \times \frac{-2k\cdot l}{k^2l^2+\delta^2} \tag{11.4.26}$$

$$\mathrm{Im}\, L^{122}_{\mu\beta}(k,k') = -\mathrm{Im}\, L^{211}_{\mu\beta}(k,k') = \mathrm{Im}\, \Lambda^{122}_{\mu\beta}(k,k') \tag{11.4.27}$$

In the function L_μ we defined the following functions :

$$N_\mu = (4m^2 + l^2)\gamma_\mu - 4ml_\mu \ ,$$

$$N_\mu(\varepsilon) = 2m(\varepsilon - k_0)\gamma^0\gamma_\mu + 4m^2\gamma_\mu + 2\not{r}\gamma_\mu\gamma^0(\varepsilon - k_0)$$
$$- 2m\gamma_\mu\not{l} + 2m\gamma\cdot l\gamma_\mu - 4\not{n}_j g^j{}_\mu - 2\not{r}\gamma\cdot l\gamma_\mu \ ,$$

$$\sigma_\mu(\varepsilon) = N_\mu(\varepsilon)\delta(k_0 - l_0 - \varepsilon) + N_\mu(-\varepsilon)\delta(k_0 - l_0 + \varepsilon) \ ,$$

$$\sigma'_\mu(\varepsilon') = N'_\mu(\varepsilon')\delta(k'_0 - l_0 - \varepsilon') + N'_\mu(-\varepsilon')\delta(k'_0 - l_0 + \varepsilon') \ ,$$

$$N'_\mu(\varepsilon') = 2m\gamma_\mu\gamma^0(\varepsilon' - k'_0) + 4m^2\gamma_\mu + 2(\varepsilon' - k'_0)\gamma^0\gamma_\mu\not{l}$$
$$- 2m\not{r}\gamma_\mu + 2m\gamma_\mu\gamma\cdot l - 4\not{n}_j g^j{}_\mu - 2\gamma_\mu\gamma\cdot l\not{l} \ ,$$

$$\sigma_\mu(\varepsilon,\varepsilon') = [N_\mu(\varepsilon,\varepsilon')\delta(k'_0 - l_0 - \varepsilon') + N_\mu(\varepsilon,-\varepsilon')\delta(k'_0 - l_0 + \varepsilon')]\delta(k_0 - l_0 - \varepsilon)$$
$$+ [N_\mu(-\varepsilon,\varepsilon')\delta(k'_0 - l_0 - \varepsilon') + N_\mu(-\varepsilon,-\varepsilon')\delta(k'_0 - l_0 + \varepsilon')]\delta(k_0 - l_0 + \varepsilon) \ ,$$

$$N_\mu(\varepsilon,\varepsilon') = -2m\gamma_\mu\gamma^0(\varepsilon' - k'_0)\gamma^0\gamma_\mu\gamma^0(\varepsilon - k_0) + 2m\gamma_\mu\gamma^0(\varepsilon' - k'_0)\gamma^0\gamma_\mu$$
$$+ 2m(\varepsilon - k_0)\gamma^0\gamma_\mu + 4m^2\gamma_\mu - 2m\gamma\cdot l\gamma_\mu(\varepsilon - k_0)\gamma^0 - 2(\varepsilon' - k'_0)\gamma^0\gamma_\mu\gamma\cdot l$$
$$+ 4ml_j g^j{}_\mu - 2\gamma\cdot l\gamma_\mu\gamma\cdot l \ .$$

(11.4.28)

Electron Propagator in External Field

In the presence of an external field $A_\mu(x)$, the Dirac equation is modified to read

$$(i\not{D} - m)\psi(x) = 0 \qquad (12.1.1)$$

where $\not{D} = \not{\partial} + ie\not{A}$ is the covariant derivative. In this case, the electron propagator is not spatially invariant (i.e. it is not a function of the difference x - y). We can express the electron propagator in the presence of an external field in terms of a complete set of solutions of (12.1.1). denoting by u the "positive energy" solutions and by v the "negative energy" solutions, they are of the form :

$$u_a(x) = u_a(x)e^{-iE_a t} \quad (E_a > 0) \qquad (12.1.2a)$$
$$v_b(x) = v_b(x)e^{+iE_b t} \quad (E_b > 0) \qquad (12.1.2b)$$

It must be stressed that solutions of the type (12.1.2) which are periodic in time exist only for static external fields. We therefore assume that $A_\mu = A_\mu(x)$.

The solution to (12.1.1) can now be expressed as

$$\psi(x) = \sum_r [\, u_r(x)a_r e^{-iE_r t} + v_r(x)b_r^* e^{iE_r t}] \qquad (12.1.3)$$

Here the operators a_r and b_r are not the annihilation operators but are related to them through the Bogoliubov transformation :

$$a_r = c(E_r)a_r(\beta) + d(E_r)\tilde{a}_r^\dagger(\beta) \qquad (12.1.4a)$$
$$\tilde{a}_r = c(E_r)\tilde{a}_r(\beta) - d(E_r)a_r^\dagger(\beta) \qquad (12.1.4b)$$
$$b_r = c(E_r)b_r(\beta) - d(E_r)\tilde{b}_r^\dagger(\beta) \qquad (12.1.5a)$$
$$\tilde{b}_r = c(E_r)\tilde{b}_r(\beta) + d(E_r)b_r^\dagger(\beta) \qquad (12.1.5b)$$

Defining the two-point function by

$$S^{\alpha\beta}(x-y) = \langle 0(\beta)|T\psi^{\alpha}(x)\bar{\psi}^{\beta\dagger}(y)|0(\beta)\rangle \qquad (12.1.6)$$

from which we quickly find :

$$\begin{aligned} S^{\alpha\beta}(x,y) = &\sum_{r} [u_r(x)\bar{u}_r(y)\{U(E_r)\tau\Theta(t_x-t_y)U^{\dagger}(E_r)\}_{\alpha\beta}e^{-iE_r(t_x-t_y)}] \\ &+ \sum_{r} [v_r(x)\bar{v}_r(y)\{U(-E_r)\tau\Theta(t_x-t_y)U^{\dagger}(-E_r)\}_{\alpha\beta}e^{iE_r(t_x-t_y)}] \end{aligned} \qquad (12.1.7)$$

where

$$\Theta(t_x-t_y) = \begin{pmatrix} \theta(t_x-t_y) & 0 \\ 0 & \theta(t_y-t_x) \end{pmatrix} \qquad (12.1.8)$$

The Fourier transform of (12.1.7) is defined by :

$$S^{\alpha\beta}(x,y) = \frac{1}{(2\pi)^4}\int d^4p\,d^4p'\, S^{\alpha\beta}(p,p')e^{-ip\cdot x + ip'\cdot y} \qquad (12.1.9)$$

When calculating the Lamb shift, (12.1.7) will be of interest to us in the non-relativistic case and for the one particle theory (i.e. electrons only or positrons only). For the non-relativistic case, we want to relate the spinors $u_r(x)$ or $v_r(x)$ to the scalar Schrodinger function $\varphi(x)$* . This reduction of the spinors to a scalar function is accomplished by separating the spinors into large and small components. Considering $u(x)$ ($v(x)$ follows by analogy), we write :

$$u = u_l + u_s \qquad (12.1.10)$$

where

$$u_l = \frac{1}{2}(1-\gamma^0)u \qquad (12.1.11a)$$

$$u_s = \frac{1}{2}(1+\gamma^0)u \qquad (12.1.11b)$$

---

* Here we also disregard the interaction between orbital and spin angular momentum.

We can now break the equation

$$Hu = Eu$$

satisfied by (12.1.2a) into a pair of equations for u_l and u_s. Thus we write

$$\begin{aligned} H &= i\gamma^0\gamma\cdot\nabla + e\gamma^0 A + m\gamma^0 \\ &= K + V + m\gamma^0 \end{aligned} \tag{12.1.12}$$

where

$$K = i\gamma^0\gamma\cdot(\nabla + ieA) + V + m\gamma^0 \tag{12.1.13}$$

$$V = e\gamma^0 A_0 \tag{12.1.14}$$

so that we get

$$Ku_s = (E - V - m)u_l \tag{12.1.15a}$$

$$Ku_l = (E - V + m)u_s \tag{12.1.15b}$$

From this it is evident that in the non-relativistic limit we can neglect the small components. Thus we can replace u by u_l and in addition we can omit the factors γ^0 since $\gamma^0 u_l = u_l$. Since $1/2(1 - \gamma^0)$ is a projection operator of rank two, we can reduce the 4 component spinors to two components. In other words, we can write

$$u_l(x) = \varphi(x)\chi \tag{12.1.16}$$

where χ is a spin function normalized so that $\chi\bar{\chi} = 1$. Furthermore, we also have the relationship :

$$\sum_{r=1}^{2} \chi_r\bar{\chi}_r = 1 \tag{12.1.17}$$

We can now express (12.1.7) as :

$$S_{NR}(x,y) = \sum_n \varphi_n(x)\varphi_n^*(y)e^{-iE_n(t_x - t_y)}U(E_n)\tau\Theta(t_x - t_y)U^\dagger(E_n) \tag{12.1.18}$$

The Fourier transform of (12.1.18) is given by

$$S_{NR}(p,p') = \frac{1}{(2\pi)^4}\int d^4x d^4x' \, e^{ip\cdot x - ip'\cdot x}\sum_n \varphi_n(x)\varphi_n^*(x')e^{-iE_n(t-t')}\, U(E_n)\tau\Theta(t-t')U^\dagger(E_n)$$

$$= \frac{1}{2\pi}\int dtdt' \, e^{i(Et - Et')}\sum_n \varphi_n(p)\varphi_n^*(p')e^{-iE_n(t-t')}U(E_n)\tau\Theta(t-t')U^\dagger(E_n)$$

(12.1.19)

where

$$\varphi_n(p) = \frac{1}{(2\pi)^{3/2}}\int d^3x \, \varphi_n(x)e^{-ip\cdot x} \tag{12.1.20}$$

Now we introduce

$$T = \frac{t+t'}{2}, \quad \zeta = t - t' \tag{12.1.21}$$

$$S_{NR}(p,p') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} d\zeta \sum_n \varphi_n(p)\varphi_n^*(p')e^{ip_0(T+\zeta/2) - ip_0'(T-\zeta/2)}e^{-iE_n\zeta}$$

$$\times U(E_n)\tau\Theta(\zeta)U^\dagger(E_n)$$

$$= \int_{-\infty}^{\infty} d\zeta\delta(p_0 - p_0')\sum_n \varphi_n(p)\varphi_n^*(p')e^{i(p_0+p_0')\zeta/2}e^{-iE_n\zeta}U(E_n)\tau\Theta(\zeta)U^\dagger(E_n)$$

$$= i\delta(p_0 - p_0')\sum_n \varphi_n(p)\varphi_n^*(p')U(E_n)\tau\begin{pmatrix} \frac{1}{p_0 - E_n + i\delta} & 0 \\ 0 & \frac{-1}{p_0 - E_n - i\delta} \end{pmatrix}U^\dagger(E_n)$$

$$= i\delta(p_0 - p_0')U(p_0)\sum_n \varphi_n(p)\varphi_n^*(p')\tau\begin{pmatrix} \frac{1}{p_0 - E_n + i\delta} & 0 \\ 0 & \frac{-1}{p_0 - E_n - i\delta} \end{pmatrix}U^\dagger(p_0)$$

(12.1.22)

We rewrite this as

$$S_{NR}(p,p') = 2\pi\delta(p_0 - p_0')U(p_0)\sum_n \varphi_n(\mathbf{p})\varphi_n^*(\mathbf{p}')\tau\begin{pmatrix}\delta_+(p_0 - E_n) & 0\\ 0 & \delta_-(p_0 - E_n)\end{pmatrix}U^\dagger(p_0) \tag{12.1.23}$$

where

$$\delta_+(p_0 - E_n) = \frac{i}{2\pi}\frac{\mathcal{P}}{p_0 - E_n} + \frac{1}{2}\delta(p_0 - E_n) \tag{12.1.24a}$$

$$\delta_-(p_0 - E_n) = -\frac{i}{2\pi}\frac{\mathcal{P}}{p_0 - E_n} + \frac{1}{2}\delta(p_0 - E_n) \tag{12.1.24b}$$

Separating the propagator into a temperature dependent and independent part gives :

$$S_{NR(0)}(p,p') = 2\pi\delta(p_0 - p'_0)\sum_n \varphi_n(\mathbf{p})\varphi_n^*(\mathbf{p}')\begin{pmatrix}\delta_+(p_0 - E_n) & 0\\ 0 & \delta_-(p_0 - E_n)\end{pmatrix} \tag{12.1.25a}$$

$$S_{NR(\beta)}(p,p') = -2\pi\delta(p_0 - p'_0)\frac{\delta(p_0 - E_n)}{e^{\beta p_0} + 1}\sum_n \varphi_n(\mathbf{p})\varphi_n^*(\mathbf{p}')\begin{pmatrix}1 & e^{\beta p_0/2}\\ e^{\beta p_0/2} & -1\end{pmatrix} \tag{12.1.25b}$$

In addition to the above results we have the following : the free field equation in thermal doublet notation is given by :

$$\Lambda^{\alpha\beta}(\partial)\psi^\alpha(x) = 0 \tag{12.1.26}$$

where $\Lambda(\partial) = (i\not{\partial} - m)$ multiplied by the unit matrix. When we operate with $\Lambda(\partial)$ from the left on (12.1.9), we get

$$\Lambda^{\alpha\beta}(\partial)S^{\beta\gamma}(x,x') = \frac{1}{(2\pi)^4}\int d^4p d^4p' \,(\not{p} - m - e\not{A})S^{\beta\gamma}(p,p')e^{-ip\cdot x + ip'\cdot x'} \tag{12.1.27}$$

This we rewrite as

$$\Lambda^{\alpha\beta}(\partial)S^{\alpha\beta}(x,x') = \frac{1}{(2\pi)^4}\int d^4q d^4p d^4p' \,[(\not{p} - m)\delta(p-q) - \not{V}(p-q)] \times S^{\alpha\beta}(q,p')e^{-ip\cdot x + ip'\cdot x'} \tag{12.1.28}$$

where

$$V_\mu(p) = -\frac{e}{(2\pi)^{3/2}} A_\mu(\mathbf{p})\delta(p^0) \tag{12.1.29}$$

$$A_\mu(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int A_\mu(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\, d^3p \tag{12.1.30}$$

The expression in square brackets in the integrand of (12.1.28) is taken to be the matrix element of an integral operator in momentum space

$$(\not{p} - m)\delta(p-q) - \not{V}(p-q) = \langle p|\hat{\not{p}} - \hat{\not{V}} - m|q\rangle$$

We make a similar interpretation for $S(p,p')$ so that in operator notation

$$S(p,p') = \hat{U}\langle p|\hat{S}|p'\rangle\hat{U}^\dagger \tag{12.1.31}$$

and

$$\hat{S} = -\hat{U}(\not{p} - m - \hat{\not{V}})^{-1}\hat{U}^\dagger \tag{12.1.32}$$

When an external field is present we modify the Feynman diagrams by drawing the electron lines as two solid lines as shown in fig. 12.1.

Fig. 12.1 Electron-electron scattering in an external field.

The S-matrix is also modified in the presence of an external field. It can be developed iteratively as in the absence of an external field. When the external field is present the interaction Hamiltonian is given by

$$H_{int} = H_r + H_e \tag{12.1.33}$$

where H_r is the interaction of the electron with the photon and H_e is the interaction of the electron with the external field. In this way the nth order term of the S-matrix is a polynomial of order n in the external field and we write this as :

$$S^{(n)} = \sum_{v=0}^{n} S^{(nv)} \tag{12.1.34}$$

where $S^{(nv)}$ contains all the terms of vth power in the external field A_μ. Thus for example, $S^{(n0)}$ describes the system in the absence of an external field and $S^{(n1)}$ contains the external field to first power only. The radiative corrections of certain processes (i.e. Coulomb scattering) are described by $S^{(31)}$ terms. The terms $S^{(nv)}$ can be illustrated diagrammatically by introducing a $\times$ at every corner where the external field acts. This is illustrated in fig. 12.2.

Fig. 12.2 The external field acting at a corner.

Radiative Corrections to Coulomb Scattering and the Lamb Shift

Radiative Corrections to Coulomb scattering

When considering the elastic scattering of electrons by atoms, the electron cloud surrounding the electron is neglected. Hence the atom is considered as an infinitely heavy positive charge of magnitude Ze. The problem is then reduced to the motion of an electron in a Coulomb field. Coulomb scattering is defined then as the scattering of an electron to all orders in the Coulomb field, but to zero order in the radiation field. The radiative corrections to Coulomb scattering considers the effect of the presence of virtual (or real) photons emitted and reabsorbed by the scattered electron. The Coulomb scattering diagram to second order is illustrated in fig. 13.1. The radiative corrections to Coulomb scattering are illustrated in fig. (13.2).

Fig. 13.1 Coulomb scattering.

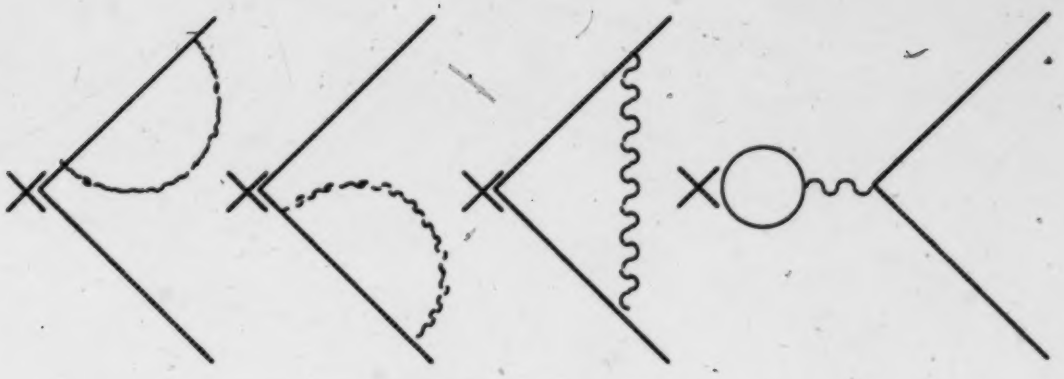


Fig. 13.2 Radiative corrections to Coulomb scattering.

When evaluating the S-matrix elements of diagrams, the Feynman rules are modified slightly. These modifications are evident when one uses the reduction formulas for S-matrix elements. Considering the case of fermions, the reduction formula extended to TFD is :

$$\langle \text{out}|\cdots d^{\alpha'}_{\text{out}}(q')\cdots b^{\alpha}_{\text{out}}(q_1)b^{\dagger\beta}_{\text{in}}(k_1)\cdots d^{\dagger\beta'}_{\text{in}}(k')\cdots|\text{in}\rangle$$

$$= \text{disc.} + (-iZ^{-1/2})^n(iZ^{-1/2})^{n'}\int d^4x_1\cdots d^4y'_1\cdots \exp[-i\Sigma(k\cdot x + k'\cdot x' - q\cdot y - q'\cdot y')]\cdot$$

$$\times \bar{u}^{\alpha}(q_1)(i\vec{\partial}_{y_1} - m)\cdots\bar{v}^{\beta'}(k'_1)(i\vec{\partial}_{x'_1} - m)\langle 0(\beta)|T\cdots\overline{\psi}^{\alpha'}(y'_1)\cdots\psi^{\alpha}(y_1)\overline{\psi}^{\beta}(x_1)\cdots\psi^{\beta'}(x'_1)\cdots|0(\beta)\rangle$$

$$\times (-i\overleftarrow{\partial}_{x_1} - m)u^{\beta}(k_1)\cdots(-i\overleftarrow{\partial}_{y'_1} - m)v^{\alpha'}(q'_1)\cdots \tag{13.1.1}$$

In this expression, disc. means all those diagrams that have no external lines. The constant Z is the wave function renormalization constant. Also the operators d_{out} , b_{out} are the annihilation operators in the plain wave expansion of the wave function of the outgoing particles while d_{in} , b_{in} are those for the incoming particles. The vacuum expectation value gives the thermal propagators as discussed earlier. However, since

$$(i\vec{\partial} - m)S^{\alpha\beta}(x,x') = -\delta^{\alpha\beta}\delta(x - x') \tag{13.1.2a}$$

$$S^{\alpha\beta}(x,x')(-i\overleftarrow{\partial} - m) = -\delta^{\alpha\beta}\delta(x - x') \tag{13.1.2b}$$

then when we integrate out the delta functions, the whole procedure has the effect of replacing the propagators of the external lines by the wave functions of the incoming or outgoing particle. Also, there is an integration over the coordinate at each corner or over the internal momenta in momentum space. Now we note that :

$$\bar{u}^1 = \bar{u} \text{ and } \bar{u}^2 = \tilde{\bar{u}}^{\dagger t} = \tilde{\bar{u}}^{*\dagger t} = \bar{u}^{\dagger\dagger} = \bar{u}$$
$$u = u \text{ and } u^2 = \tilde{u}^{\dagger t} = u^{*\dagger t} = u^{\dagger\dagger} = u$$

Thus, we will ignore the thermal indices of the wave functions of the external lines. In a similar fashion we find that for external photon lines the propagator is replaced by the photon polarization vector, denoted by e, in the evaluation of S-matrix elements.

With this in mind, the radiative corrections to Coulomb scattering give the following matrix element :

$$\langle p'|M^{(31)}|p\rangle = \frac{e}{\sqrt{2\pi}} \frac{m}{\sqrt{EE'}} \bar{u}(\mathbf{p}') \sum_{\beta\gamma\delta} Q_\mu^{\beta\gamma\delta}(p,p') A^\mu(\mathbf{q}) u(\mathbf{p}) \qquad (13.1.3)$$

where

$$Q_\mu^{\beta\gamma\delta}(p,p') = \Sigma^{\delta\gamma}(p') S^{\gamma\beta}(p') \gamma_\mu + \gamma_\mu S^{\delta\gamma}(p) \Sigma^{\gamma\beta}(p) + \Lambda_\mu^{\beta\gamma\delta}(p,p') - \Delta^{\beta\gamma}(q) \gamma^\nu \Pi_{\nu\mu}^{\gamma\delta}(q) \qquad (13.1.4)$$

Instead of (13.1.3) we prefer to write :

$$\langle p'|M^{(31)}|p\rangle^{\beta\gamma\delta} = \frac{e}{\sqrt{2\pi}} \frac{m}{\sqrt{EE'}} \bar{u}(\mathbf{p}') Q_\mu^{\beta\gamma\delta}(p,p') A^\mu(\mathbf{q}) u(\mathbf{p}) \qquad (13.1.5)$$

We separate this into a temperature independent and dependent part :

$$\langle p'|M^{(31)}|p\rangle^{\beta\gamma\delta} = \langle p'|M^{(31)}|p\rangle_0^{\beta\gamma\delta} + \langle p'|M^{(31)}|p\rangle_\beta^{\beta\gamma\delta} \qquad (13.1.6)$$

The temperature independent part is given by :

$$\langle p'|M^{(31)}|p\rangle^{111} = \langle p'|M^{(31)}|p\rangle^{222} = -\frac{e}{\sqrt{2\pi}}\frac{m}{\sqrt{EE'}}\frac{\alpha}{3\pi}\left[\frac{q^2}{m^2}\left(\ln\frac{m}{\lambda} - \frac{3}{8} - \frac{1}{5}\right)\right. \tag{13.1.7}$$
$$\left. \times \bar{u}(p')\gamma_\mu A^\mu(q)u(p) - \frac{3}{4im}\bar{u}(p')\gamma_\mu A^\mu(q)\not{q}u(p)\right]$$

The above result is evaluated in the case of small momentum transfers :

$$q^2 = q^2 << m^2$$

Also, λ in (13.1.7) is the photon mass and is introduced as a cutoff.

The temperature dependent part of (13.1.6) is expressed in terms of :

$$\begin{aligned} Q^{\beta\gamma\delta}_{\mu(\beta)}(p,p') &= [\Sigma^{\gamma\delta}_{(0)}(p')S^{\gamma\delta}_{(\beta)}(p') + \Sigma^{\gamma\delta}_{(\beta)}(p')S^{\beta\gamma}(p')]\gamma_\mu \\ &+ \gamma_\mu[S^{\gamma\delta}(p)\Sigma^{\beta\gamma}_{(\beta)}(p) + S^{\gamma\delta}_{(\beta)}(p)\Sigma^{\beta\gamma}_{(0)}(p)] \\ &+ L^{\beta\gamma\delta}_{\mu(\beta)}(p,p') + \Lambda^{\beta\gamma\delta}_{\mu f(\beta)}(p,p') \\ &- \Delta^{\beta\gamma}_{(0)}(q)\gamma^\nu\Pi^{\gamma\delta}_{\nu\mu(\beta)}(q) - \Delta^{\beta\gamma}_{(\beta)}(q)\gamma^\nu\Pi^{\gamma\delta}_{\nu\mu}(q) \end{aligned} \tag{13.1.8}$$

In other words :

$$\langle p'|M^{(31)}|p\rangle^{\beta\gamma\delta}_{(\beta)} = \frac{e}{\sqrt{2\pi}}\frac{m}{\sqrt{EE'}}\bar{u}(p')Q^{\beta\gamma\delta}_{\mu(\beta)}(p,p')A^\mu(q)u(p) \tag{13.1.9}$$

The Lamb Shift

The experiments of Lamb and Retherford were the first decisive experiments to measure the energy level separation between the $2^2S_{1/2} - 2^2P_{1/2}$ states of hydrogen and deuterium. It was a great achievement of theoretical physics that the results of quantum electrodynamics (QED) agreed remarkably well with the experimental results. This shift in the $2^2S_{1/2}$ and $2^2P_{1/2}$ states is due to the radiative corrections which are neglected in the Dirac theory, and is referred to as the Lamb shift. The very accurate experimental

measurements are motivation for extending the results of QED to the case of finite temperature. This is what we do next.

The lowest order radiative corrections ΔE to a level E in a hydrogen like atom is given by :

$$\Delta E = \frac{1}{2\pi i} \langle E|M|E\rangle \qquad (13.2.1)$$

The diagrams for this correction in the presence of an external field are illustrated in fig. 13.3 :

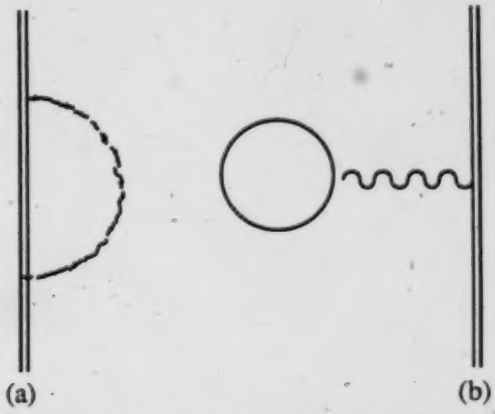


Fig. 13.3 The lowest order radiative corrections to the motion of an electron in an external field. Diagram (a) is the fluctuation diagram and (b) the polarization diagram.

The fluctuation diagram is due to the virtual emission and reabsorption of photons while the polarization diagram is due to the polarization of the vacuum in the presence of an external field. We consider the fluctuation diagram first.

The S-matrix element for the fluctuation diagram is :

$$\Delta E_F^{\alpha\beta} = \frac{ie^2}{(2\pi)^4} \int \bar{u}_0 \gamma_\mu S^{\alpha\beta}(p - k - V) \gamma^\mu u_0 \Delta^{\alpha\beta}(k)\, d^4k \qquad (13.2.2)$$

where

$$S^{\alpha\beta}(p-k-V) = [U(p_0)(\not{p}-\not{k}-\not{V}-m)^{-1}U^{\dagger}(p_0)]^{\alpha\beta} \qquad (13.2.3)$$

Here the subscript zero refers to the energy level E_0 whose corrections are being sought. The operator V is defined in terms of the external field as :

$$Vu = -\frac{e}{(2\pi)^{3/2}}\int \not{A}(p-p')u(p')d^3p' \qquad (13.2.4)$$

In the expression (13.2.2) there is contained the self energy of the free electron. This will have to be subtracted so that we get the observable level shift. In other words, the level shift is the difference between the self energy of the bound and free electron. As was mentioned previously, there is no closed form for the propagator in the external field. We must use approximations for further progress. First we use the following identity :

$$\begin{aligned}(\not{p}-\not{k}-\not{V}-m)^{-1} = (\not{p}-\not{k}-m)^{-1} + (\not{p}-\not{k}-m)^{-1}\not{V}(\not{p}-\not{k}-m)^{-1} \\ + (\not{p}-\not{k}-m)^{-1}\not{V}(\not{p}-\not{k}-\not{V}-m)^{-1}\not{V}(\not{p}-\not{k}-m)^{-1}\end{aligned} \qquad (13.2.5)$$

Now, since $U(\varepsilon)$ is a unitary matrix, we can expand (13.2.3) as

$$\begin{aligned}&U(p_0)(\not{p}-\not{k}-\not{V}-m)^{-1}U^{\dagger}(p_0) = U(p_0)(\not{p}-\not{k}-m)^{-1}U^{\dagger}(p_0) \\ &+ U(p_0)(\not{p}-\not{k}-m)^{-1}U^{\dagger}(p_0)\not{V}U(p_0)(\not{p}-\not{k}-m)^{-1}U^{\dagger}(p_0) \\ &+ U(p_0)(\not{p}-\not{k}-m)^{-1}U^{\dagger}(p_0)\not{V}U(p_0)(\not{p}-\not{k}-\not{V}-m)^{-1}U^{\dagger}(p_0)\not{V}U(p_0)(\not{p}-\not{k}-m)^{-1}U^{\dagger}(p_0)\end{aligned} \qquad (13.2.6)$$

With this identity, the matrix element can be expressed in terms of diagrams as shown in fig (13.4).

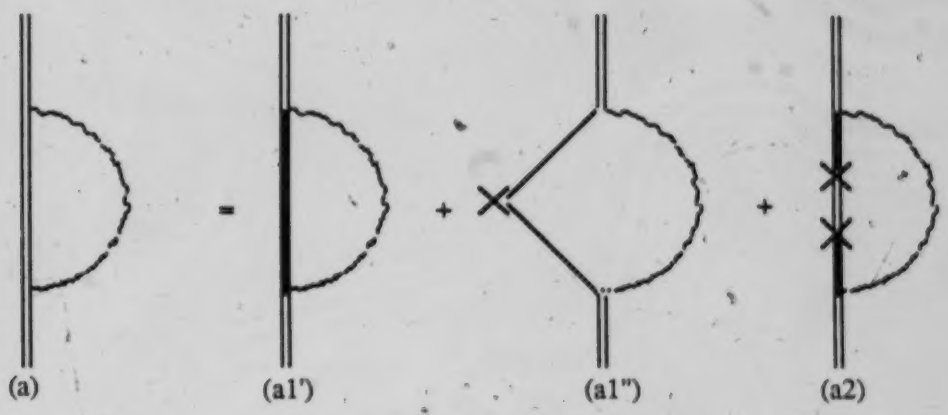


Fig. 13.4 Decomposition of the fluctuation diagram.

In this diagram, the external double lines indicate Coulomb wave functions and the internal double lines indicate Coulomb propagation functions. Diagrams (a1) are called the one potential part and (a2) the many potential part. Although the separation in fig. 13.4 is exact, we must make approximations for further progress. The first diagram (a1') will be relevant in the cancellation of infinite terms.

The second diagram (a1") gives a factor α from the emission and reabsorption of the virtual photon; a factor αZ from the single action of the Coulomb field in the intermediate state.; and a factor a^{-3} where a is the atomic radius which is of the order of magnitude of the normalization of the Coulomb wave function. For hydrogen like atoms :

$$|\psi_n^{NR}(0)|^2 = \frac{1}{\pi a_n^3} = \frac{(Z\alpha m)^3}{\pi n^3}$$

where n is the principal quantum number. Thus the diagram (a1") gives an energy correction of order of magnitude $\alpha(\alpha Z)^4 m$ times the corresponding integral. Now we know that the vertex part contains an infrared divergence and since it is of a logarithmic type, the expectation value of the vertex will give a logarithm. This divergence is expected to be compensated from terms arising by diagram (a2). However the functional form of the

logarithmic factor will not disappear. In the zero temperature theory this factor is of the order of v^2 where v is the electron's velocity. In our case v is of the order of αZ. Thus diagram (a1") should contain terms of order $\alpha(\alpha Z)^4 \ln \alpha Z$ and $\alpha(\alpha Z)^4$.

Diagram (a1) is to be evaluated relativistically and (a2) non-relativistically. We consider first the contribution of the one potential part ΔE_1. It consists of the fluctuation and the polarization diagrams :

$$\Delta E_1 = \Delta E_{F1} + \Delta E_{P1}$$

The diagrams for ΔE_{F1} are (a1') and (a1") and the free electron self energy must be subtracted from them. We do this by expanding the external lines of (a1') in the Born approximation. Thus assuming the electron to be at rest, we approximate the Coulomb function by

$$u = u_f + (\not{p} - m)^{-1} \not{V} u \qquad (13.2.7)$$

Then the diagram (a1') is separated as illustrated in fig. 13.5.

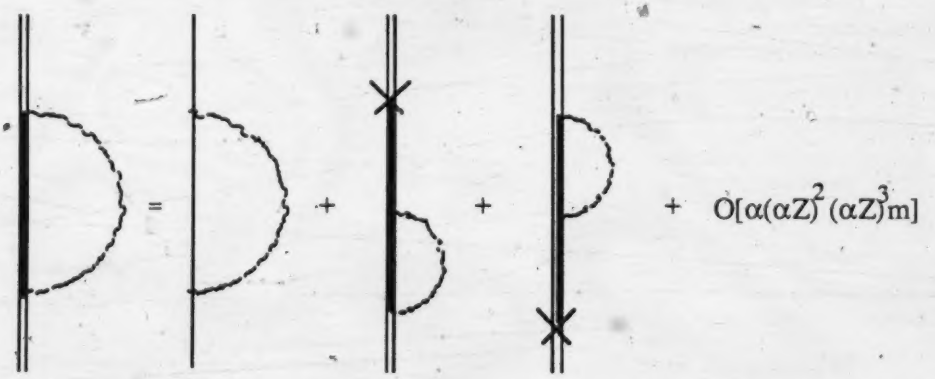


Fig. 13.5 Approximate decompisation of diagram (a1').

The first diagram is simply the free electron self energy which we subtract. Thus we are left only with the other two diagrams in fig. 13.5. These are seen to be the same as the radiative corrections to Coulomb scattering. The only difference is that diagrams 13.5

have Coulomb wave functions for the external lines while diagrams (13.2) have plain waves for the external lines. The results obtained in section (13.1) are valid for free initial and final states irrespective of the momentum, which includes momenta of order αZm that occur in bound states. So we can take the results (13.1.7) and (13.1.9) and evaluate it for Coulomb wave function. This corresponds to an expansion of the Coulomb wave functions in terms of plain waves so that the error is of the order of the binding energy $(\alpha Z)^2 m$ which is negligible. Making the same arguments for the polarization diagram we separate ΔE_1 as :

$$\Delta E_1 = \Delta E_{1(0)} + \Delta E_{1(\beta)}$$

The temperature independent part is given by :

$$\langle p'|M^{(31)}|p\rangle^{111} = \langle p'|M^{(31)}|p\rangle^{222} = -\frac{ie}{(2\pi)^{3/2}}\left[\frac{\alpha}{3\pi}\left(\ln\frac{m}{\lambda} - \frac{3}{8} - \frac{1}{5}\right)\right.$$
$$\left.\times \int \left(\bar{u}_0(\mathbf{p}')\frac{q^2}{m^2}\not{A}(q)u_0(\mathbf{p}) + \frac{\alpha}{4\pi}\,\bar{u}_0(\mathbf{p}')\not{A}(q)\frac{i\not{q}}{m}u_0(\mathbf{p})\right) d^3p'd^3p\right] \tag{13.2.8}$$

It can be written in coordinate space as

$$\Delta E_1 = \frac{\alpha}{3\pi}\left(\ln\frac{m}{\lambda} - \frac{3}{8} - \frac{1}{5}\right)\langle 0(\beta)|\frac{\nabla^2}{m^2}V|0(\beta)\rangle$$
$$- \frac{i\alpha}{4\pi}\langle 0(\beta)|\gamma\cdot\frac{\nabla}{m}V|0(\beta)\rangle \tag{13.2.9}$$

The temperature dependent part is given by :

$$\langle p'|M^{(31)}|p\rangle_{(\beta)}^{\beta\gamma\delta} = \frac{ie}{(2\pi)^{3/2}}\int \bar{u}_0(\mathbf{p}')Q_{\mu(\beta)}^{\beta\gamma\delta}(p,p')A^{\mu}(q)u_0(\mathbf{p})\,d^3p'd^3p \tag{13.2.10}$$

Now we discuss the many potential part ΔE_2. This is given by :

$$\Delta E_2^{\alpha\beta} = \frac{e^2}{(2\pi)^4}\sum_{pol.}\int \bar{u}_0 \not{\varepsilon}\left[U(\varepsilon)\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\delta\tau}U^{\dagger}(\varepsilon)\not{V}U(\varepsilon)(\not{p}-\not{k}-\not{V}-m)^{-1}U^{\dagger}(\varepsilon)\right. \tag{13.2.11}$$

$$\left. \times \not{V}U(\varepsilon)\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\delta\tau}U^{\dagger}(\varepsilon)\right]_{\alpha\beta}\not{\varepsilon}u_0\Delta^{\alpha\beta}(k)d^4k$$

Here the sum is an average over the initial spin states and a sum over the final spin states. We simplify ΔE_2 by taking into account all permissible approximations to order $\alpha(\alpha Z)^4 \ln\alpha Z$ and $\alpha(\alpha Z)^4$. Also we assume a small photon mass λ so that

$$\Delta_0(k) = \frac{\tau}{k^2-\lambda^2+i\delta\tau}$$

With this in mind, we note that the photon's momentum **k** will contribute mainly when it is of the same order of magnitude as the photon energy ω, that is $O((\alpha Z)^2 m)$. Thus we can neglect **k** in the numerator of the free propagator because it is of order $(\alpha Z)^2$ compared with m. Also, the electron's momentum is of order $\alpha Z m$ and its energy $m - O((\alpha Z)^2 m)$. Then since p^2+m^2 vanishes in the denominator of the free particle propagator and k^2 is of order $(\alpha Z)^4 m^2$, we can write

$$(p-k)^2 - m^2 \approx -2p\cdot k = -2m\omega$$

This approximation for the denominator is permissible only for the zero temperature part of the free propagator. For the temperature dependent part we do not ignore the photons' momentum, because otherwise we get $\delta(\omega)$ type terms which will give a divergent result. Thus in this approximation we can write

$$S_{(0)}(p) = \frac{\not{p}+m}{-2m\omega+i\delta\tau}$$

$$S_{(\beta)}(p) = \frac{2\pi i(\not{p}+m)}{e^{\beta m}+1}\delta[(m-\omega)^2-\varepsilon]\begin{pmatrix} -1 & e^{\beta m/2} \\ e^{\beta m/2} & 1 \end{pmatrix} \tag{13.2.12}$$

Furthermore, since we are to evaluate ΔE_2 in the non-relativistic limit we can use the non-relativistic Coulomb propagation function (12.1.25). This means that we can replace it by the sum over the large components of the electron's functions $u_l\bar{u}_l$ and ignore the positron states which involve transitions of order $(\alpha Z)^2 m/2m$ smaller. Thus we write (12.1.26,27) as

$$S_{(NR)(0)}(p-k\,,\,p'-k) \to 2\pi\delta(E_0-E')\sum_n u_n(\mathbf{p}-\mathbf{k})\bar{u}_n(\mathbf{p}'-\mathbf{k}) \times \begin{pmatrix} \delta_+(E_n-E_0+\omega) & 0 \\ 0 & \delta_-(E_n-E_0+\omega) \end{pmatrix} \tag{13.2.13a}$$

$$S_{(NR)(\beta)}(p-k\,,\,p'-k) \to -2\pi\delta(E_0-E')\sum_n u_n(\mathbf{p}-\mathbf{k})\bar{u}_n(\mathbf{p}'-\mathbf{k})\frac{\delta(E_n-E_0+\omega)}{e^{\beta E_0}+1} \times \begin{pmatrix} 1 & e^{\beta E_0/2} \\ e^{\beta E_0/2} & -1 \end{pmatrix} \tag{13.2.13b}$$

Here we have introduced the total energy of the state u_0, $m - E_0$ and that of u_n, $m - E_n + \omega$. Furthermore since $\mathbf{k}$ is of order αZ compared with $\mathbf{p}$, we ignore it in the argument of u_n.

The zero temperature part of ΔE_2 is then given by

$$\Delta E^{\alpha\beta}_{2(0)} = \frac{\alpha}{2\pi^2}\sum_{pol.}\int \bar{u}_0 \not{\epsilon}\left\{\frac{\not{p}+m}{-2m\omega+i\delta\tau}\gamma\sum_n\begin{pmatrix} \delta_+(E_n-E_0+\omega) & 0 \\ 0 & \delta_-(E_n-E_0+\omega) \end{pmatrix}u_n\bar{u}_n\gamma \times \frac{\not{p}+m}{-2m\omega+i\delta\tau}\right\}_{\alpha\beta}\not{\epsilon}\bar{u}_0\left\{\frac{\tau\, d^4k}{k^2-\lambda^2+i\delta\tau}\right\}_{\alpha\beta} \tag{13.2.14}$$

For the sum over polarizations we use the result that the emission probability of a longitudinal meson of momentum $\mathbf{k}$, mass λ is

$$\frac{\lambda^2}{\mathbf{k}^2+\lambda^2}$$

times the emission probability of a transverse photon. With this we write

$$\Delta E^{\alpha\beta}_{2(0)} = \frac{\alpha}{2\pi^2}\frac{1}{12m^2}\sum_n \int d^3k \int d\omega \left[\begin{pmatrix} \delta_+(E_n - E_0 + \omega) & 0 \\ 0 & \delta_-(E_n - E_0 + \omega) \end{pmatrix} \frac{1}{(\omega^2 - i\delta\tau)} \right]_{\alpha\beta} \times (2 + \lambda^2/\omega^2) \left(\frac{\tau}{(-k^2 + \omega^2 - \lambda^2 + i\delta\tau)} \right)_{\alpha\beta} |\langle 0|\gamma(\not{p} + m)\not{y}|n\rangle|^2 \tag{13.2.15}$$

The expectation value is simplified to

$$|\langle 0|\gamma(\not{p} + m)\not{y}|n\rangle| = 2(E_0 - E_n)\langle 0|p|n\rangle$$

Then after a simple integration over angles, we get

$$\Delta E^{\alpha\beta}_{2(0)} = \frac{4\alpha}{3\pi m^2}\sum_n \int_0^\infty |k|^2 d|k| \int_{-\infty}^{\infty} \left[\begin{pmatrix} \delta_+(E_n - E_0 + \omega) & 0 \\ 0 & \delta_-(E_n - E_0 + \omega) \end{pmatrix} \frac{1}{(\omega^2 - i\delta\tau)} \right]_{\alpha\beta} \times (2 + \lambda^2/\omega^2)\,(E_0 - E_n)^2 |\langle 0|p|n\rangle|^2 \left(\frac{\tau}{(-k^2 + \omega^2 - \lambda^2 + i\delta\tau)} \right)_{\alpha\beta} d\omega \tag{13.2.16}$$

For the ω-integration of the 1-1 component of (13.2.16) we close the contour by a large half circle in the negative imaginary part of the ω-plane. For the 2-2 component it is closed in the positive imaginary part. Then since the energy level is given by the diagonal sum, we find

$$\Delta E_{2(0)} = -\frac{i8\alpha}{3m^2}\sum_n (E_0 - E_n)^2 |\langle 0|p|n\rangle|^2 \int_0^\infty \frac{k^2 d|k|}{2\omega_0^3}\left(1 + \frac{\lambda^2}{\omega_0^2}\right) \times [\delta_+(E_0 - E_n + \omega_0) + \delta_-(E_0 - E_n - \omega_0)] \tag{13.2.17}$$

where $\omega_0 = \sqrt{k^2 + \lambda^2}$. The Lamb shift is given by the real part of (13.2.17), the imaginary part gives the line width. Thus, performing the remaining integration in (13.2.17) gives the real part (in the limit $\lambda \to 0$)

$$\Delta E_{2(0)} = \frac{4\alpha}{3\pi m^2} \sum_n (E_n - E_0) |\langle 0|p|n\rangle|^2 \left(\ln \frac{\lambda}{2|E_0 - E_n|} + \frac{5}{6} \right) \qquad (13.2.18)$$

When we use the well known result

$$\sum_n |\langle n|p|0\rangle|^2 (E_n - E_0) = \frac{1}{2} \langle 0|\nabla^2 V|0\rangle \qquad (13.2.19)$$

we can rewrite (13.2.19) as

$$\Delta E_{2(0)} = \frac{\alpha}{3\pi} \left(\ln \frac{\lambda}{2k_0} + \frac{5}{6} \right) \left\langle 0 \left| \frac{\nabla^2}{m^2} V \right| 0 \right\rangle \qquad (13.2.20)$$

where

$$\ln \frac{k_0(n_0,l)}{Z^2 Ry} = \frac{\sum_n |\langle n|p|n_0 l\rangle|^2 (E_n - E_0) \ln(|E_n - E_0|/Ry)}{\sum_n |\langle n|p|n_0 0\rangle|^2 (E_n - E_0)} \qquad (13.2.21)$$

The energy k_0 is an average excitation energy. The quantum numbers n_0 and l are associated with E_0.

Similarly we can calculate the temperature dependent parts of (13.2.11), although we cannot always solve the d^3k or the ω-integrations. Also we do not separate the real and imaginary parts. We discuss the results below

$$\Delta E^{\alpha\beta}_{2(\beta 1)} = -\frac{\alpha}{2\pi^2}\sum_{pol.}\int d^4k\, \bar{u}_0 \not{\epsilon}\left\{\frac{\not{p}+m}{-2m\omega+i\delta\tau}\not{y}\sum_n \frac{\delta(E_n-E_0+\omega)}{e^{\beta E_0}+1}\begin{pmatrix} 1 & e^{\beta E_0/2} \\ e^{\beta E_0/2} & -1\end{pmatrix} u_n\bar{u}_n \not{y} \times \frac{\not{p}+m}{-2m\omega+i\delta\tau}\right\}_{\alpha\beta} \not{\epsilon} u_0 \left\{\frac{\tau}{k^2-\lambda^2+i\delta\tau}\right\}_{\alpha\beta} \tag{13.2.22}$$

The ω integration is the same, so that when we add the 1-1 and the 2-2 components, we get

$$\Delta E_{(\beta 1)} = -\frac{8i\alpha}{3m^2}\sum_n (E_n - E_0)^2 |\langle n|p|0\rangle|^2 \frac{1}{e^{\beta E_0}+1} \times \int_0^\infty \frac{k^2 d|k|}{2\omega_0^3}\left(1+\frac{\lambda^2}{2\omega_0^2}\right)\delta(E_n - E_0 + \omega_0) \tag{13.2.23}$$

Similarly, other temperature dependent terms are :

$$\Delta E_{2(\beta 2)} = \frac{4i\alpha}{3m^2}\sum_n (E_n - E_0)^2 |\langle n|p|0\rangle|^2 \int_0^\infty \frac{k^2 d|k|}{2\omega_0^3}\left(1+\frac{\lambda^2}{2\omega_0^2}\right)\frac{1}{e^{\beta\omega_0}-1} \times [\delta(E_n - E_0 + \omega_0) - \delta(E_n - E_0 - \omega_0)] \tag{13.2.24}$$

Since, E_n - E_0 is slowly varying, we can approximate it by E_1 - E_0. Then evaluating the sum, we get

$$\Delta E_{2(\beta 2)} = \frac{4i\alpha}{3m^2}\langle 0|[V, p]|0\rangle \int_0^\infty \frac{k^2 d|k|}{2\omega_0^3}\left(1+\frac{\lambda^2}{2\omega_0^2}\right)\frac{1}{e^{\beta\omega_0}-1} \times [\delta(E_1 - E_0 + \omega_0) - \delta(E_1 - E_0 - \omega_0)] \tag{13.2.25}$$

Applying the same arguments that lead to (13.2.25) give for for the next temperature dependent term :

$$\Delta E_{2(\beta 3)} = -\frac{8i\alpha}{3\pi^2 m^2}\frac{\langle 0|[V\,,\,\mathbf{p}]|0\rangle}{e^{\beta m}+1}\int_0^\infty k^2 d|k|\left\{\frac{1}{(m-\varepsilon)[((m-\varepsilon)^2-\omega_0^2]}\frac{\mathcal{P}}{E_1-E_0}\left(1+\frac{\lambda^2}{2(m-\varepsilon)^2}\right)\right.$$
$$\left.+\left(1+\frac{\lambda^2}{2\omega_0^2}\right)\frac{1}{2m(4m^2-\omega_0^2)}\frac{\mathcal{P}}{E_1-E_0+2m}\left(1+\frac{\lambda^2}{8m^2}\right)\right\} \tag{13.2.26}$$

In addition we get the following terms :

$$\Delta E_{2(\beta 4)} = -\frac{8\pi\alpha}{3m}\frac{\langle 0|[V\,,\,\mathbf{p}\,]|0\rangle}{e^{\beta m}+1}\int_0^\infty\int_{-\infty}^\infty k^2 d|k|\left(1+\frac{\lambda^2}{2\omega^2}\right)$$
$$\times\frac{1}{e^{\beta|\omega|}-1}\left[\frac{\delta_-(E_1-E_0+\omega)}{\omega+i\delta}-\frac{\delta_+(E_1-E_0+\omega)}{\omega-i\delta}\right]\delta[(p_0-\omega)^2-\varepsilon^2]\delta(\omega^2-\omega_0^2)\,d\omega \tag{13.2.27}$$

$$\Delta E_{2(\beta 5)} = +\frac{8i\alpha}{3m}\sum_n (E_n-E_0)^2|\langle n|\mathbf{p}|0\rangle|^2\frac{1}{e^{\beta m}+1}\int_0^\infty\int_{-\infty}^\infty\frac{k^2 d|k|}{\omega(\omega^2-\omega_0^2)}\frac{2i\delta}{\omega^2+\delta^2}$$
$$\times\left(1+\frac{\lambda^2}{2\omega^2}\right)\delta(E_n-E_0+\omega)\delta[(p_0-\omega)^2-\varepsilon^2]\frac{e^{\beta(E_0+m)/2}-1}{e^{\beta E_0}+1}\,d\omega \tag{13.2.28}$$

$$\Delta E_{2(\beta 6)} = +\frac{16\pi\alpha}{3m}\sum_n (E_n-E_0)^2|\langle n|\mathbf{p}|0\rangle|^2\frac{1}{e^{\beta m}-1}\int_0^\infty\int_{-\infty}^\infty k^2 d|k|\left(1+\frac{\lambda^2}{2\omega^2}\right)\frac{2\omega}{\omega^2+\delta^2}$$
$$\times\frac{1}{e^{\beta|\omega|}-1}\,\delta(E_n-E_0+\omega)\delta[(E_0-\omega)^2-\varepsilon^2]\delta(\omega^2-\omega_0^2)\frac{e^{\beta(E_0+m)/2}-1}{e^{\beta E_0}+1}\,d\omega \tag{13.2.29}$$

$$\Delta E_{2(\beta 7)} = \frac{4i\alpha}{3m^2}\sum_n\frac{(E_n-E_0)^2|\langle n|\mathbf{p}|0\rangle|^2}{e^{\beta E_0}+1}\int_0^\infty\frac{k^2 d|k|}{2\omega_0^3}\left(1+\frac{\lambda^2}{2\omega_0^2}\right)\frac{1}{e^{\beta\omega_0}-1}$$
$$\times\,[\delta(E_n-E_0+\omega_0)+\delta(E_n-E_0-\omega_0)] \tag{13.2.30}$$

$$\Delta E_{2(\beta 8)} = -\frac{4\alpha}{3\pi}\sum_n \frac{e^{\beta m}-1}{e^{\beta m}+1}(E_n - E_0)^2 |\langle n|p|0\rangle|^2 \int d^4k \left(1+\frac{\lambda^2}{2\omega^2}\right) \times \frac{\delta_\pm(E_n - E_0 + \omega)}{k^2 - \lambda^2}\{\delta[(m-\omega)^2 - \varepsilon^2]\}^2 \tag{13.2.31}$$

$$\Delta E_{2(\beta 9)} = -\frac{8i\alpha}{3}\sum_n \frac{(E_n - E_0)^2 |\langle n|p|0\rangle|^2}{e^{\beta m}+1}\int d^4k \left(1+\frac{\lambda^2}{2\omega^2}\right) \times \frac{\delta(E_n - E_0 + \omega)}{e^{\beta\omega}-1}\{\delta[(m-\omega)^2 - \varepsilon^2]\}^2 \delta[\omega^2 - \omega_0^2] \tag{13.2.32}$$

$$\Delta E_{2(\beta 10)} = -\frac{4\alpha}{3\pi}\sum_n (E_n - E_0)^2 |\langle n|p|0\rangle|^2 \int d^4k \left(1+\frac{\lambda^2}{2\omega^2}\right) \times \frac{\delta(E_n - E_0 + \omega)}{k^2 - \lambda^2 \pm i\delta}\{\delta[(m-\omega)^2 - \varepsilon^2]\}^2 \frac{2 - 4e^{\beta(m+E_0)/2} - 2e^{\beta m}}{(e^{\beta E_0}+1)(1+e^{\beta m})^2} \tag{13.2.33}$$

So, the temperature dependent correction to ΔE_2 is given by the sum of the real part of equations (13.2.23-13.2.33). The zero temperature terms (13.2.9) and (13.2.21) can be combined to give the usual zero temperature result :

$$\Delta E_{(0)} = \frac{8Z^4}{n^3}\frac{\alpha^3}{3\pi}\left[\ln\frac{m}{2k_0(n,0)} + \frac{19}{30}\right] Ry \qquad (l=0)$$
$$\Delta E_{(0)} = \frac{8Z^4}{n^3}\frac{\alpha^3}{3\pi}\left[\ln\frac{Z^2 Ry}{k_0(n,l)} + \frac{3}{8}\frac{c_{lj}}{2l+1}\right] Ry \qquad (l\neq 0) \tag{13.2.34}$$

where

$$c_{lj} = \begin{cases} \dfrac{1}{l+1} & \text{for } j = l + \dfrac{1}{2} \\ -\dfrac{1}{l} & \text{for } j = l - \dfrac{1}{2} \end{cases} \tag{13.2.35}$$

Summary

We have presented a review of equilibrium thermo field dynamics. Today, it is just one of several approaches to a field theory at finite temperature [26]. Ojima has shown that TFD is equivalent to several of these approaches [27]. Of all these approaches, TFD is perhaps the most useful for practical calculations because the Feynman diagram method can still be used. Furthermore, the renormalization method, the renormalization group and the Ward-Takahashi identities are similar to the usual quantum field theory. TFD can be considered as a straightforward extension of the usual QFT to finite temperature, and only a few modifications of the operator formalism from that at zero temperature are required. An advantage that TFD has over the conventional field theory is that generalized annihiliation operators can be defined. This enables us to use the generalized Feynman diagram method.

In thermo field dynamics, a new field, the tilde field, is introduced that doubles the number of degrees of freedom, and can control the temperature of the system. With this doubling, the results of zero temperature QFT can be recast in terms of the thermal doublet notation. The need for the doubling of the degrees of freedom can be intuitively understood as follows : when a system has non-zero temperature, it contains some number of thermally excited quanta. This introduces the dynamical freedom to annihilate (or create the holes of) these thermally excited quanta in addition to the freedom of the usual excitation process.

In contrast to other thermal field theories, TFD can treat thermal phenomena in a closed system. Since it can do so for any such system, it is suitable for the analysis of the thermal development of the universe and also to analyze the thermal behavior of high energy reactions such as the quark-gluon plasma in QCD.

We have used TFD to calculate the temperature corrections to the famous Lamb shift. The Lamb shift will always be remembered as one of the great achievements of the twentieth century. With it began a new age in physics, that of quantum field theory and renormalization. Recently, with the many finite temperature field theories, it has become possible to extend the calculation of the Lamb shift to finite temperature. In particular, because TFD is a straightforward extension of conventional QFT, it is the most useful for extending the Lamb shift calculation. This is achieved in this thesis.

The temperature corrections to the Lamb shift are given by the sum of the real 111 and the 222 components of (13.2.9). This is due to the radiative corrections to Coulomb scattering. In addition the this sum, we must add the real parts of (13.2.23-13.2.33). These terms are due to the motion of the electron in the Coulomb field. The calculation does give the correct zero temperature terms. However, that is only a slight assurance for the calculation being correct. For complete confidence in the calculation, we must check to see that it agrees with experiment. Since it is difficult with these expressions, to estimate the magnitude of the temperature effect, we must carry out a numerical calculation of all the equations mentioned above. Also, a high and low temperature limit of these results would be very interesting. Only then can we be sure that the results are in agreement with experimental measurements. This remaining test of the calculation will be achieved in a different work.

In addition to the Lamb shift, there is another important quantity in quantum electrodynamics; the magnetic moment of the electron. This is related to the Lamb shift problem and must be taken into account. This has been achieved by several people [28,29,30,31,32]. However, any one result among theirs does not agree with any other. We will pursue this problem in another work.

Bibliography

Claude Itzykson and Jean-Bernard Zuber, Quantum Field Theory , (McGraw-Hill Inc, 1980).

James D. Bjorken and Sidney D. Drell, Relativistic Quantum Mechanics , (McGraw-Hill Inc, 1964).

James D. Bjorken and Sidney D. Drell, Relativistic Quantum fields , (McGraw-Hill Inc, 1965).

H. Umezawa, H. Matsumoto and M. Tachiki, Thermo Field Dynamics and Condensed states , (North Holland : Amsterdam, 1982).

J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons , (Second corrected printing, Springer-Verlag, 1980).

H. Matsumoto, "Thermo field dynamics and its development", in Progress in Quantum Field Theory , ed. H. Ezawa and S. Kamefuchi (North-Holland : Amsterdam, 1986).

H. Matsumoto, "The Causal Function in Many Body Problems", *Fortschritte der phys.* **25**, (1977) 1.

H. Umezawa, An Essay on Thermal Physics , preprint (to be published in physics essays).

H. Umezawa and Y. Yamanaka, Micro, Macro and Thermal Concepts in Quantum Field Theory , preprint.

H. Umezawa, What is Thermal , preprint

H. Matsumoto, I. Ojima and H. Umezawa, "Pertubration and Renormalization in Thermo Field Dynamics", *Ann. of Phys.* **152**, (1984) 348.

Selected Papers on Quantum Electrodynamics , ed. Julian Schwinger (Dover Publications Inc, 1958).

Abraham Pais, Inward Bound , (Oxward University Press, 1986).

References

[1] R. Haag, "Concepts in quantum field theory", in Progress in Quantum Field Theory, ed. H. Ezawa and S. Kamefuchi (North-Holland : Amsterdam, 1986).

[2] H. Umezawa and Y. Yamnaka, "Micro, Macro and Thermal Concepts in Quantum Field Theory", Preprint.

[3] K.O. Friedrichs, Mathematical Aspects of the Quantum Theory of Fields, (Interscience, New York 1953).

[4] A.S. Wightman and S.S Schweber, "Configuration Space Methods in Relativistic Quantum Field Theory. I", *Phys. Rev.* **98** (1955) 812 ; L. Garding and A.S. Wightman, *Proc. Nat. Acad. Sci. USA* , **40** (1954) 617.

[5] H. Lehmann, K. Symanzik and W. Zimmerman, *Nuovo Cim.* **6** (1957) 319.

[6] T. Matsubara, *Progr. Theor. Phys.* **14** (1955) 351.

[7] A.A Abrikosov, L.P. Gorkov and I.E. Dzyaloshinskii, *Soviet Phys. JETP*, **9** (1959) 636.

[8] H. Ezawa, Y. Tomozawa and H. Umezawa, *Nuovo Cimento* **5** (1957) 810.

[9] L. Leplae, F. Mancini and H. Umezawa, "Derivation and Application of the Boson Method in Superconductivity", *Phys. Rep.* **10C** (1974) 151.

[10] Y. Takahashi and H. Umezawa, "Thermo Field Dynamics", *Collect. Phenom.* **2** (1975) 55.

[11] Willis E. Lamb, Jr. and Robert C. Retherford, "Fine Structure of the Hydrogen Atom by a Microwave Method", *Phys. Rev.* **72** (1947) 241.

[12] S. Pasternack, Letter to the Editor : "Note on the Fine Structure of H_α and D_α", *Phys. Rev.* **54** (1938) 1113.

[13] E. Uehling, "Polarization Effects in the Positron Theory" , *Phys. Rev.* **48** (1935) 55.

[14] I. Waller, *Zeitschr. f. Phys.* **62** (1930) 673.

[15] J. R. Oppenheimer, "Note on the theory of the Interaction of Field and Matter", *Phys. Rev.* **35**, (1930) 461.

[16] H.A. Bethe, "The Electromagnetic Shift of Energy Levels", *Phys. Rev.* **72**, (1947) 339.

[17] T. A. Welton, "Some Observable Effects of the Quantum-Mechanical Fluctuations of the Electromagnetic Field", *Phys. Rev.* **74** (1948) 1157.

[18] U. Fano, *Rev. Mod. Phys.* **29** (1957) 74 ; J. A. Crawford, *Nuovo Cim.* **10** (1958) 698 ; M. Schmutz, "Real-Time Green's Functions in Many Body Problems", *Z. Phys.* **B30** (1978) 97.

[19] T. Armitsu and H. Umezawa, "A General Formulation of Non-Equilibrium Thermo Field Dynamics", *Prog. Theor. Phys.* **74** (1985) 429 ; "Non-Equilibrium Thermo Field Dynamics", *Prog. Theor. Phys.* **77** (1987) 32.

[20] H. J. Kreuzer and C. G. Kuper, "On the Approach to Equilibrium in a Gibbs Ensemble", *Collect. Phen.* **2** (1975) 141.

[21] H. Umezawa, "A Role of Thermal State Conditons in Equilibrium and Non-equilibrium Thermo Field Dynamics", in Quantum Field Theory , ed. F. Mancini, (Elsevier Science Publishers B.V., 1986).

[22] H. Matsumoto, Y. Nakano and H. Umezawa, "Tilde Substitution Law in Thermo Field Dynamics: Thermal State Conditions", *Phys. Rev D* . **31** number 2 (1985) 429.

[23] R. Kubo, *J. Phys. Soc. Jap.* **12** (1957) 570 ; P. Martin and J. Schwinger, "Theory of Many Body Problems", *Phys. Rev.* **115**, (1959) 1342.

[24] M. Suzuki, "Quantum fluctuations, Thermo Field Dynamics and Quantum Monte Carlo Methods", in Progress in Quantum Field Theory , ed. H. Ezawa and S. Kamefuchi (North-Holland, 1986).

[25] E. E. Salpeter and H. A. Bethe, "A Relativistic Equation for Bound State Problems", *Phys. Rev.* **84** number 6 (1951) 1232.

[26] N. P. Landsman and Ch. G. van Weert, "Real- and Imaginary-Time Field Theory at Finite Temperature and Density", *Phys. Reports* , **145** , (1987) 143.

[27] I. Ojima, "Gauge Fields at Finite Temperature-'Thermo Field Dynamics' and the KMS Condition and their Extension to Gauge Theories", *Ann. Phys.* **137**, (1981) 1.

[28] F. Ruiz Ruiz and R. F. Alvarez-Estrada, "Thermo Field Dynamics and One-Loop Order QED at Finite Temperature", *Z. Phys. C - Particles and Fields* **34**, (1987) 131.

[29] G. Peressuti and B.S. Skagerstam, *Phys. Lett.* **110B**, (1982) 406.

[30] Y. Fujimoto and J.H. Yee, "The Electron Magnetic Moment at High Temperature", *Phys. Lett.* **114B**, (1982) 359.

[31] P.H. Cox and W.S. Hellman and A. Yildiz, "Finite Temperature Corrections to Field Theory: Electron Mass and Magentic Moment, and Vacuum Energy", *Ann. Phys.* **154**, (1984) 211.

[32] A. Palanques-Mestre, R. Tarrach, "Electrons in a Heated Vacuum", *Lett. Nuovo. Cimento* **43**, (1985) 237.

Appendix A

Free Fields for Physical Particles

The Free Field Equation

We require that the free physical field be a linear superposition of the annihilation and creation operators $\alpha(\mathbf{k}; \mathbf{x}, t)$ and $\alpha^{\dagger}(\mathbf{k}; \mathbf{x}, t)$ with coefficients that depend neither on $\mathbf{x}$ nor on t. The time and space development of these operators is given by

$$\alpha(\mathbf{k}; \mathbf{x}, t) = \alpha(\mathbf{k})\exp[i\{\mathbf{k}\cdot \mathbf{x} - \omega(\mathbf{k})t\}] \tag{A1.1}$$

$$\alpha^{\dagger}(\mathbf{k}; \mathbf{x}, t) = \alpha^{\dagger}(\mathbf{k})\exp[-i\{\mathbf{k}\cdot \mathbf{x} - \omega(\mathbf{k})t\}] \tag{A1.2}$$

A further requirement of the construction of the free physical field is that there exist a projection mechanism which can project out each annihilation or creation operator. This guarantees that the dynamical map can be expressed in terms of products of free physical fields.

The free physical field satisfies a homogeneous differential equation of the the form

$$\lambda(\partial)\varphi^{0}(x) = 0 \tag{A1.3}$$

Free fields can be classified in two ways. When

$$\lambda(\partial) = i\frac{\partial}{\partial t} - \varepsilon(\nabla) \tag{A1.4}$$

then (A.3) is refers to as an equation of type 1. On the other hand, it is an equation of type 2 when

$$\lambda(\partial) = \frac{\partial^2}{\partial t^2} + \omega^2(\nabla) \tag{A1.5}$$

The Divisor

The free field equation suggests that there should exist a differential operator $d(\partial)$ such that for a type 1 equation

$$d(\partial)\lambda(\partial) = i\frac{\partial}{\partial t} - \varepsilon(\nabla) \tag{A2.1}$$

and

$$d(\partial)\lambda(\partial) = -\left[\frac{\partial^2}{\partial t^2} + \omega^2(\nabla)\right] \tag{A2.2}$$

for equations of type 2. If we denote by $\Delta_G(x)$ the Green's function for either of equations (A2.1) and (A2.2) then we can show that

$$\lambda(\partial)d(\partial)\Delta_G(x) = \delta(x)\delta(t) \tag{A2.3}$$

so that $d(\partial)\Delta_G(x)$ is the Green's function for the free field equation (A1.3). The operator $d(\partial)$ is called the divisor.

The Hermitization Matrix

When we make the following definition

$$\lambda(p)\exp[i(\mathbf{p}\cdot\mathbf{x} - p_0 t)] = \lambda(\partial)\exp[i(\mathbf{p}\cdot\mathbf{x} - p_0 t)] \tag{A3.1}$$

then the eigenvalue equation $\lambda(p)u = 0$ for some vector u has the eigenvalues $p_0 = \pm\omega(\mathbf{p})$. Since the eigenvalues are real, $\lambda(p)u = 0$ should be equivalent to an eigenvalue equation of a certain hermitian matrix with the same eigenvalues. Thus we are lead to the existence of a non-singular matrix η which makes $\eta\lambda(p)$ hermitian. This is expressed as

$$\lambda^\dagger(p)\eta^\dagger = \eta\lambda(p) \tag{A3.2}$$

or as

$$\lambda^\dagger(-\partial)\eta^\dagger = \eta\lambda(\partial) \tag{A3.3}$$

From the equations (A2.1) and (A2.2) we get the result that $d(p)\eta^{-1}$ is also hermitian. The matrix η is called the hermitization matrix. We now introduce the following notation

$$\bar{\varphi}^0(x) = \varphi^{0\dagger}(x)\eta \tag{A3.4}$$

so that the hermitian conjugate of the free field equation becomes

$$\bar{\varphi}^0(x)\lambda(-\overleftarrow{\partial}) = 0 \tag{A3.5}$$

The Lagrangian for the Free Field

The Lagrangian for the free field is given by

$$\mathfrak{L}^0_\varphi = \int d^4x\, \bar{\varphi}^0(x)\lambda(\partial)\varphi^0(x) \tag{A4.1}$$

The Inner Product

To find the expression for the inner product of wave functions, we assume that $\lambda(\partial)$ is a polynomial of degree less then or equal to 2 in $\partial/\partial t$. In this case, we write $\lambda(\partial)$ as

$$\lambda(\partial) = \lambda^{(0)}(\nabla) + i\lambda^{(1)}(\nabla)\frac{\partial}{\partial t} + \lambda^{(2)}(\nabla)\frac{\partial^2}{\partial t^2} \tag{A5.1}$$

From this definition, we define an operator Γ by

$$\overleftrightarrow{\Gamma} = \lambda^{(1)}(\nabla) - i\lambda^{(2)}(\nabla)\overleftrightarrow{\frac{\partial}{\partial t}} \tag{A5.2}$$

where

$$\overleftrightarrow{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} - \overleftarrow{\frac{\partial}{\partial t}} \tag{A5.3}$$

Then when f(x) and g(x) satisfy the free field equation, their inner product is given by

$$\int d^3x\, \bar{f}(x)\overleftrightarrow{\Gamma} g(x) \tag{A5.4}$$

The Structure of the Free Field

For a type 2 equation, the free field has the structure as given by

$$\varphi^0(x) = \sum_r \int d^3k\, [u^r(k)\alpha^r(k)\exp(i\{k\cdot x - \omega(k)t\}) + v^r(k)\beta^{r\dagger}(k)\exp(-i\{k\cdot x - \omega(-k)t\})] \tag{A6.1}$$

whereas for a type 1 equation, it has the structure

$$\varphi^0(x) = \sum_r \int d^3k\, \{\theta[\varepsilon(k)]u^r(k)\alpha^r(k) + \theta[-\varepsilon(-k)]v^r(-k)\beta^{r\dagger}(-k)\}e^{ik\cdot x - i\varepsilon(k)t} \tag{A6.2}$$

Here, $\theta(x)$ is the step function. The amplitudes $u^r(k)$ and $v^r(k)$ satisfy the orthonormalization conditions

$$\bar{u}^r(k)\Gamma[k, \omega(k)]u^s(k) = (2\pi)^{-3}\delta_{rs}\hbar \tag{A6.3}$$

$$\bar{v}^r(k)\Gamma[-k, \omega(-k)]u^s(k) = -(2\pi)^{-3}\rho\delta_{rs}\hbar \tag{A6.4}$$

where

$$\Gamma(k, E) = \lambda^{(1)}(k) - 2E\lambda^{(2)}(k) \tag{A6.5}$$

Sum Rules

The plane waves

$$u^r(k, x) = u^r(k)\exp[-i\{k\cdot x - \omega(k)t\}] \tag{A7.1}$$

$$v^r(k, x) = v^r(k)\exp[-i\{k\cdot x - \omega(-k)t\}] \tag{A7.2}$$

satisfy the following sum rules

$$\sum_r \int d^3k\, u^r(k, x)\bar{u}^r(k, x') = i\hbar d(\partial)\Delta^+(x - x') \tag{A7.3}$$

$$\sum_r \int d^3k\, v^r(k, x)\bar{v}^r(k, x') = -i\rho\hbar d(\partial)\Delta^-(x - x') \tag{A7.4}$$

where for equations of type 2

$$\Delta^{\pm}(x) = \mp\, i(2\pi)^{-3}\int \frac{d^3k}{2\omega(\pm k)} \exp\{\pm i[k\cdot x - \omega(\pm k)t]\} \tag{A7.5}$$

with

$$\Delta^{+}(-x) = -\Delta^{-}(x) \tag{A7.6}$$

$$\delta(t)\frac{\partial}{\partial t}\Delta^{\pm}(x) = -\frac{1}{2}\delta(x)\delta(t) \tag{A7.7}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2(\nabla)\right) \tag{A7.8}$$

For equations of type 1, it can be shown that (A7.3) is satisfied if

$$\Delta^{+}(x) = -i(2\pi)^{-3}\int d^3k\, e^{i(k\cdot x - \omega t)} \tag{A7.9}$$

with

$$\delta(t)\Delta^{+}(x) = -i\delta(x)\delta(t) \tag{A7.10}$$

$$\left(i\frac{\partial}{\partial t} - \omega(\nabla)\right)\Delta^{+}(x) = 0 \tag{A7.11}$$

Commutation Relations and Statistics

From the structure of the free fields, we can readily show that

$$[\varphi_j^0(x)\,,\,\overline{\varphi}_k^0(y)]_{\pm} = i\hbar d_{jk}(\partial)[\Delta^{+}(x-y) + \rho\rho'\Delta^{-}(x-y)] \tag{A8.1}$$

where $\rho = +1(-1)$ for fermion (boson) fields and

$$\rho' = \begin{cases} 0 & \text{for type-1 equation with } +\varepsilon(k) \\ +1 & \text{for type-2 boson equations} \\ -1 & \text{for type-1 fermion equations} \end{cases} \tag{A8.2}$$

When we introduce the causality requirement for operators F_i by writing

$$[\,F_i(x,t)\,,\,F_j(y,t)] = 0 \qquad \text{for } x \neq y \tag{A8.3}$$

we find that it is satisfied when and only when

$$[\varphi_j^0(x)\ ,\ \bar{\varphi}_k^0(y)]_{\pm}\delta(t_x - t_y) = 0 \quad \text{for } x \neq y \tag{A8.4}$$

For type-1 equations, the causality requirement is automatically satisfied. For type-2 equations, on the other hand, we obtain from (A7.5)

$$\Delta^+(x-y) + \Delta^-(x-y) = 0 \qquad t_x = t_y \tag{A8.5a}$$

$$\Delta^+(x-y) - \Delta^-(x-y) = -\frac{i}{(2\pi)^3}\int \frac{d^3k}{\omega(k)}\, e^{ik\cdot(x-y)} \qquad t_x = t_y \tag{A8.5b}$$

so that the causality requirement gives us

$$\rho\rho' = 1 \tag{A8.6}$$

Now when we introduce the following function

$$\Delta(x-y) = \Delta^+(x-y) + \Delta^-(x-y) \tag{A8.7}$$

Then we can write

$$[\varphi^0(x)\ ,\ \bar{\varphi}^0(y)]_{\pm} = i\hbar d(\partial)\Delta(x-y) \tag{A8.8}$$

for type-2 equations, and

$$[\varphi^0(x)\ ,\ \bar{\varphi}^0(y)]_{\pm} = i\hbar d(\partial)\Delta^+(x-y) \tag{A8.9}$$

for type-1 equations.

From these considerations we obtain

$$\rho = \begin{cases} +1 & \text{for boson fields} \\ -1 & \text{for fermion fields} \end{cases} \tag{A8.10}$$

which gives the result that the causality requirement uniquely determines the statistics of the particles described by field equations of type-2.

Appendix B

Product Rules of Two Point Functions

In many situations, one encounters diagrams (for example the self energy to second order in which there is a product of two point functions. For convenience we list here the results of such products.

When we encounter a product of boson two point functions of the form as given by (6.3.4) then we write the product as follows :

$$\int \frac{dl_0}{2\pi}\left[U_B(\kappa_+)\,\tau\left(l_0+\frac{k_0}{2}-\kappa_+ + i\delta\tau\right)^{-1}U_B(\kappa_+)\right]_{\alpha\beta}$$
$$\times\left[U_B(\kappa_-)\,\tau\left(l_0-\frac{k_0}{2}\mp(\kappa_- - i\delta\tau)\right)^{-1}U_B(\kappa_-)\right]_{\beta\alpha}$$
$$= i\int d\kappa\,\sigma(\kappa;\kappa_+,\kappa_-)\,[U_B(\kappa)\,\tau\{k_0-\kappa+i\delta\tau\}^{-1}U_B(\kappa)]_{\alpha\beta} \qquad \text{(B1.1)}$$

where

$$\sigma(\kappa;\kappa_+,\kappa_-) = \delta(\kappa-\kappa_+\pm\kappa_-)\,[\,f_B(\kappa_+)-f_B(\pm\kappa_-)] \qquad \text{(B1.2)}$$

This formula and others of its kind are derived component by component.

Similarly the product of fermion two point functions is given by

$$\varepsilon^{\alpha}\varepsilon^{\beta}\int \frac{dl_0}{2\pi}\Big[U(\kappa_+)\Big(l_0+\frac{k_0}{2}-\kappa_+ + i\delta\tau\Big)^{-1}U^{\dagger}(\kappa_+)\Big]_{\alpha\beta}$$
$$\times\Big[U(\kappa_-)\Big(l_0-\frac{k_0}{2}-\kappa_- + i\delta\tau\Big)^{-1}U^{\dagger}(\kappa_-)\Big]_{\beta\alpha}$$
$$= i\int d\kappa\, \sigma(\kappa;\kappa_+,\kappa_-)\,[U_B(\kappa)\,\tau\,\{k_0-\kappa+i\delta\tau\}^{-1}U_B(\kappa)]_{\alpha\beta} \qquad \text{(B1.3)}$$

where $\varepsilon^{\alpha}=\tau^{\alpha\alpha}$ and

$$\sigma(\kappa;\kappa_+,\kappa_-) = -\delta(\kappa-\kappa_+ +\kappa_-)\,[\,f_F(\kappa_+)-f_F(\kappa_-)] \qquad \text{(B1.4)}$$

The product of a fermion two point function and a boson two point function gives a fermion two point function. Thus when the fermion two point function is given by (6.2.6) and the boson two point function by (6.3.4) then

$$\int \frac{dl_0}{2\pi}\Big[U(\kappa_1)\{k_0-l_0-\kappa_1+i\delta\tau\}^{-1}U^{\dagger}(\kappa_1)\Big]_{\alpha\beta}$$
$$\times\Big[U_B(\kappa_2)\,\tau\,\{l_0\mp(\kappa_2-i\delta\tau)^2\}^{-1}U_B(\kappa_2)\Big]_{\alpha\beta}$$
$$= i\int_{-\infty}^{\infty} d\kappa\,\sigma(\kappa;\kappa_1,\kappa_2)\,[U(\kappa)\,\{k_0-\kappa+i\delta\tau\}^{-1}U^{\dagger}(\kappa)]_{\alpha\beta} \qquad \text{(B1.5)}$$

where

$$\sigma(\kappa;\kappa_1,\kappa_2) = -\delta(\kappa-\kappa_1\mp\kappa_2)[f_B(\pm\kappa_2)+1-f_F(\kappa_1)] \qquad \text{(B1.6)}$$

On the other hand, when the boson two point function is that of a scalar particle as given by (6.4.3) then we get

$$\int \frac{dl_0}{2\pi}\left[U(\kappa_1)\left\{k_0 - l_0 - \kappa_1 + i\delta\tau\right\}^{-1}U^{\dagger}(\kappa_1)\right]_{\alpha\beta}$$
$$\times\left[U_B(\kappa_2)\,\tau\left\{l_0^2 - (\kappa_2 - i\delta\tau)^2\right\}^{-1}U_B(\kappa_2)\right]_{\alpha\beta}$$
$$= i\int_{-\infty}^{\infty} d\kappa\,\sigma(\kappa;\kappa_1,\kappa_2)\,[U(\kappa)\,\{k_0 - \kappa + i\delta\tau\}^{-1}U^{\dagger}(\kappa)]_{\alpha\beta} \qquad \text{(B1.7)}$$

with

$$\sigma(\kappa;\kappa_1,\kappa_2) = -\frac{1}{2\kappa_2}\,[\delta(\kappa-\kappa_1-\kappa_2)\{f_B(\kappa_2)+1-f_F(\kappa_1)\}$$
$$-\delta(\kappa-\kappa_1+\kappa_2)\{f_B(-\kappa_2)+1-f_F(\kappa_1)\}] \qquad \text{(B1.8)}$$

Appendix C

Thermal Separation of Propagators

Consider the photon propagator (10.2.6) :

$$-i\Delta_c(k) = U_B(\omega)\tau[k_0^2 - (\omega - i\delta\tau)^2]^{-1}U_B(\omega) \qquad \text{(C.1.1)}$$

To separate this into a temperature independent and dependent part we first write out the matrix explicitly :

$$-i\Delta_c(k) = \begin{pmatrix} \dfrac{c_B^2(\omega)}{k_0 - (\omega - i\delta)^2} - \dfrac{d_B^2(\omega)}{k_0 - (\omega + i\delta)^2} & \dfrac{c_B(\omega)d_B(\omega)}{k_0 - (\omega - i\delta)^2} - \dfrac{c_B(\omega)d_B(\omega)}{k_0 - (\omega + i\delta)^2} \\ \dfrac{c_B(\omega)d_B(\omega)}{k_0 - (\omega - i\delta)^2} - \dfrac{c_B(\omega)d_B(\omega)}{k_0 - (\omega + i\delta)^2} & \dfrac{d_B^2(\omega)}{k_0 - (\omega - i\delta)^2} - \dfrac{c_B^2(\omega)}{k_0 - (\omega + i\delta)^2} \end{pmatrix} \qquad \text{(C1.2)}$$

Now when we use :

$$c_B^2(\omega) = 1 + f_B(\omega)\ ,\ d_B^2(\omega) = f_B(\omega)$$

with

$$f_B(\omega) = \frac{1}{e^{\beta\omega} - 1}$$

Then

$$-i\Delta_c(k) = \begin{pmatrix} \dfrac{1}{k_0-(\omega-i\delta)^2} & 0 \\ 0 & \dfrac{-1}{k_0-(\omega+i\delta)^2} \end{pmatrix}$$

$$+\frac{1}{e^{\beta\omega}-1}\begin{pmatrix} \dfrac{1}{k_0-(\omega-i\delta)^2}-\dfrac{1}{k_0-(\omega+i\delta)^2} & \dfrac{e^{\beta\omega/2}}{k_0-(\omega-i\delta)^2}-\dfrac{e^{\beta\omega/2}}{k_0-(\omega+i\delta)^2} \\ \dfrac{e^{\beta\omega/2}}{k_0-(\omega-i\delta)^2}-\dfrac{e^{\beta\omega/2}}{k_0-(\omega+i\delta)^2} & \dfrac{1}{k_0-(\omega-i\delta)^2}-\dfrac{1}{k_0-(\omega+i\delta)^2} \end{pmatrix} \tag{C1.3}$$

Now since

$$\frac{1}{k_0-(\omega\mp i\delta)^2} = \frac{1}{2\omega}\left[\frac{1}{k_0-\omega\pm i\delta}-\frac{1}{k_0+\omega\mp i\delta}\right] \tag{C1.4}$$

and

$$\frac{1}{k_0-\omega\pm i\delta} = \frac{P}{k_0-\omega}\mp i\pi\delta(k_0-\omega) \tag{C1.5}$$

Then we quickly get

$$-i\Delta_c(k) = \tau[k_0^2-(\omega-i\delta\tau)^2]^{-1}+\frac{-2\pi i\delta(k_0^2-\omega^2)}{e^{\beta\omega}-1}\begin{pmatrix} 1 & e^{\beta\omega/2} \\ e^{\beta\omega/2} & 1 \end{pmatrix} \tag{C1.6}$$

The above result is the same as (10.3.27a,b). The separation of the electron propagator is performed in exactly the same way, only we use (6.2.4) and (6.2.7).

Appendix D

Notation

The metric that we use is given by

$$-g^{ii} = g^{00} = 1 \tag{D1.1}$$
$$-g_{ii} = g_{00} = 1 \tag{D1.2}$$
$$g^{\mu\nu} = g_{\mu\nu} = 0 \quad \mu \neq \nu \tag{D1.3}$$

The coordinate and momenta are defined by

$$x_\mu = (t, \mathbf{x}) \tag{D1.4}$$
$$k_\mu = (\omega, \mathbf{k}) \tag{D1.5}$$

so that the dot product of two four vectors is given by

$$\mathbf{k}\cdot\mathbf{x} = k^\mu x_\mu = k_\mu x^\mu = \omega t - \mathbf{k}\cdot\mathbf{x} \tag{D1.6}$$

where $\mathbf{k}\cdot\mathbf{x}$ is the usual three dimensional dot product. We also use the slash notation as defined by

$$\not{k} = k_\mu \gamma^\mu = k^\mu \gamma_\mu \tag{D1.7}$$

We define the derivative operator by

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla_i\right) = \frac{\partial}{\partial x^\mu} \tag{D1.8}$$
$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla_i\right) = \frac{\partial}{\partial x_\mu} \tag{D1.9}$$

Furthermore, when we write $\omega(\nabla)$ etc, it is to be understood as

$$\omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} = \omega(\nabla)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{D1.10}$$

The Dirac gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{D1.11}$$

where, $\mathbb{I}$ is the 2×2 identity matrix and the σ^i are the 2×2 Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad \text{(D1.12)}$$

The Dirac gamma matrices satisfy the following relationships

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad \text{(D1.13)}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \,,\ \gamma^{0\dagger} = \gamma^0 \,,\ \gamma^{k\dagger} = -\gamma^k \qquad \text{(D1.14)}$$

$$\gamma^0\gamma^0 = +\,\mathbb{I} \,,\ \gamma^k\gamma^k = -\,\mathbb{I} \qquad \text{(D1.15)}$$

As for indices, we use the normal convention of reserving the Greek indices for 0,1,2,3 and Latin indices for 1,2,3.